Transactions of JASCOME, Vol.2 (July 2002), Paper No. 02-070319 JASCOME

AN INTERFACE RELAXATION ALGORITHM FOR COUPLING THE FINITE ELEMENT AND BOUNDARY ELEMENT METHODS

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In this paper we briefly review some of the existing domain-decomposition coupling of the finite and boundary element methods. We also summarize a general interface relaxation framework, originally developed for the solution of composite PDEs and extend two interface relaxation algorithms to higher dimensional analysis. We further present a new interface relaxation finite element/boundary element coupling algorithm, which may be implemented on a distributed parallel or sequential computer. The method overcomes some of the limitations of the existing domain decomposition coupling methods. We also investigate the convergence of the method.

Keywords: Boundary element method; Finite element method; Interface relaxation; Coupling.

1. Introduction

The finite element method (FEM) and the boundary element method (BEM) are well known as powerful numerical techniques for solving wide range of problems in applied science and engineering. Each method has its own advantages and disadvantages, so that it is desirable to develop a combined finite element/boundary element method, which makes use of their advantages and reduces their disadvantages, and to use the combined method in situations where it is appropriate. The conventional coupling methods employ an entire unified equation for the whole domain by altering the formulation of one of the methods to make it compatible with the other, see, i.e., references [1-8] not to mention many others. However, the conventional coupling methods may destroy the positive characteristics of symmetry and bandedness that originally exist in the FEM. Furthermore the implementation of the conventional coupling procedures requires a suitably integrated finite element/boundary element software environment. This requires merging two different kinds of programs together to form an integrated finite element/boundary element software environment, which will demand considerable effort.

Consequently, it may be necessary to preserve the nature of both the FEM and BEM, rather than to force any into the other format and thus the domain decomposition coupling methods have been developed [9-13]. In these methods, separate computing for the BEM and FEM sub-domains and successive renewal of the degrees of freedom on the interface of both sub-domains are performed to reach the final convergence. The domain decomposition coupling methods offer many advantages over the conventional coupling methods and appear to be promising. However, the important issue of convergence of the domain decomposition coupling methods is not fully addressed. The problem is still open for establishing the convergence conditions and for the selection of the relaxation parameters. Elleithy et al. [14,15] investigated the convergence of the sequential Dirichlet-Neumann domain decomposition-coupling method and research is underway for establishing the convergence conditions of the parallel Dirichlet-Neumann and parallel Neumann-Neumann coupling methods. Moreover, some of the existing domain decomposition coupling methods may not be applicable in problems where the Neumann boundary conditions are specified on the entire external boundary of the FEM or BEM sub-domains. The overlapping domain decomposition coupling method is capable of handling such cases [9,10]. However, the overlapping may create a serious complication in the Schwarz method, even when the global problem is having a simple geometry.

In this paper, we present an interface relaxation algorithm for coupling the FEM and BEM. The paper is arranged as follows: in Section 2, a summary of a general interface relaxation framework, originally developed for the solution of composite PDEs [20] is presented. Section 2 further extends two interface relaxation algorithms to higher dimensional analysis. Section 3 reviews various existing domain decomposition coupling methods. A new interface relaxation-coupling algorithm, which overcomes some of the limitations of the existing domain decomposition coupling methods, is presented in Section 4. Section 5 establishes the convergence conditions of the new interface relaxation coupling method. A simple numerical example is given in Section 6.

2. Interface Relaxation

Domain decomposition has proven to be an effective means of partitioning the task of solving differential equations numerically. It is mainly an algebraic approach and works by splitting the domain of the original problem into sub-domains which can be coupled in many ways. Interface

relaxation is more general than the traditional domain decomposition methods in that it allows unrelated PDE problems on different sub-domains, see references [16-20].

Global-based domain decomposition methods such as the overlapping Schwarz or the substructure-type methods are not applicable to general composite PDE problems. For simple continuity interface conditions, there may exist various Schwarz splitting-type methods, which alternatively solve Dirichlet and Neumann problems on adjacent sub-domains in a way or another. However, the interface conditions for composite PDEs may appear in a more complicated forms, thus, more general techniques are needed to handle complicated interface conditions from composite PDEs. Mu [20] presented a general framework for solving composite PDEs based on interface relaxation and it may be summarized as:

1- Denote the local PDEs by:

$$L_i u_i = f_i \text{ in } \Omega_i \text{ for } i = 1, 2, \cdots, k, \tag{1}$$

where L_i is a differential operator with the general boundary conditions specified by:

$$B_i u_i = g_i \text{ on } \Gamma_i \tag{2}$$

where B_i is a boundary condition operator and g_i is a continuous function.

2- Define Γ_{ij} as the interface between two adjacent sub-domains Ω_i and Ω_j. Let $I(i)$ be the indices of those sub-domains that are neighbors of sub-domain Ω_i. Define the boundary value problem P_i^n that is solved on Ω_i at the nth relaxation step as:

$$L_i u_i^n = f_i \text{ in } \Omega_i, \tag{3.a}$$

$$B_{ij}^n u_i^n = b_{ij}^n \text{ on } \Gamma_{ij} \text{ for } j \in I(i), \tag{3.b}$$

$$B_i u_i^n = g_i \text{ on } \Gamma_i \cap \Gamma \tag{3.c}$$

where B_{ij}^n is a boundary condition operator such that P_i^n is well posed.

The interface relaxation iteration (3) is defined on the sub-domains independently. Details of the iteration are specified by an interface handler called a relaxer. It provides for the interface Γ_{ij} to sub-domain Ω_i the right hand side data b_{ij}^n of the boundary condition according to certain relaxation procedures as well as the parameters in the definition of B_{ij}^n.

Mathematically this framework contains many existing domain decomposition methods and also allows the extension to a variety of new relaxers. We note here that it may be possible to utilize such interface relaxation methods for the coupling of FEM and BEM. Many coupling algorithms may be developed to overcome the limitations of the existing coupling ones. In the remainder of this section we will extend two interface relaxation algorithms to higher dimensional analysis.

Rice et al. [19] presented an interface relaxation algorithm for the solution of elliptic differential equations. The algorithm is classified as a one-step algorithm and it estimates a new solution for each sub-domain by solving a Dirichlet problem. The values on the interface are obtained by adding to the old ones, a geometrically weighted combination of the normal derivatives of the adjacent sub-domains. Extending the algorithm to higher dimensional analysis, it may be described as follows:

Set initial guess u_i^0 for $i = 1, 2, \ldots, k$ where k is the number of sub-domains.

Define $b_{ij}^{n+1} = u_i^n - \alpha\left(\frac{\partial u_i^n}{\partial n} + \frac{\partial u_j^n}{\partial n}\right)$ on Γ_{ij} for $j \in I(i)$ where, α is a relaxation parameter to ensure and/or accelerate convergence.

For $n = 0, 1, 2, \ldots$, do until convergence:

for $i = 1, 2, \ldots, k$ solve:

$$L_i u_i^{n+1} = f_i \text{ in } \Omega_i$$
$$B_{ij}^{n+1} u_i^{n+1} = b_{ij}^{n+1} \text{ on } \Gamma_{ij} \text{ for } j \in I(i)$$
$$B_i u_i^{n+1} = g_i \text{ on } \Gamma_i \cap \Gamma$$

where, α is a relaxation parameter to ensure and/or accelerate convergence.

The other interface relaxation algorithm is based on Robin interface conditions to transmit information across the interface, see reference [17]. The method predicts new values at the interface by making a convex combination of Dirichlet and Neumann data from the other sub-domains. Again, extending the algorithm to higher dimensional analysis, it may be described as follows:

Set initial guess u_i^0 for $i = 1, 2, \ldots, k$ where k is the number of sub-domains.

Define $b_{ij}^{n+1} = -\frac{\partial u_j^n}{\partial n} + \lambda u_j^n$ on Γ_{ij} for $j \in I(i)$ where, λ is a relaxation parameter.

For $n = 0, 1, 2, \ldots$, do until convergence:

For $i = 1, 2, \ldots, k$, solve

$$L_i u_i^{n+1} = f_i \text{ in } \Omega_i$$
$$B_{ij}^{n+1} u_i^{n+1} = b_{ij}^{n+1} \text{ on } \Gamma_{ij} \text{ for } j \in I(i)$$
$$B_i u_i^{n+1} = g_i \text{ on } \Gamma_i \cap \Gamma$$

3. Domain Decomposition Coupling Algorithms

Consider Figure 1, where the domain of the original problem is governed by Laplace equation and decomposed into FEM and BEM sub-domains, i.e., $K_i \nabla^2 u = 0 \; in \; \Omega_i$, where K_i is the material property in the sub-domain Ω_i and u is the potential. Boundary conditions are such that

the potential u, the flux $q = K\nabla u$ or their combination is prescribed at each point on the boundary.

The corresponding boundary integral equation for the BEM sub-domain is given by:

$$[H]\{u\} = [G]\{q\} \in \Gamma_B \tag{4}$$

where u and q are column matrices containing the boundary nodal values for the potential and the flux. H and G are influence coefficient matrices. For the FEM sub-domain, the assembled element equations are given by:

$$[K]\{u\} = \{f\} \in \Omega_F \tag{5}$$

where K is the stiffness matrix for the system, and u and f are the nodal potentials and integrated flux vectors respectively.

Now, let us define the following potential vectors:

u_B^I : interface potentials, approached from the BEM sub-domain

u_B^B : non-interface potentials in the BEM sub-domain

u_F^I : interface potentials, approached from the FEM sub-domain

u_F^F : non-interface potentials in the FEM sub-domain

Similarly, one can define the flux and the integrated flux vectors for the BEM and FEM sub-domains, respectively.

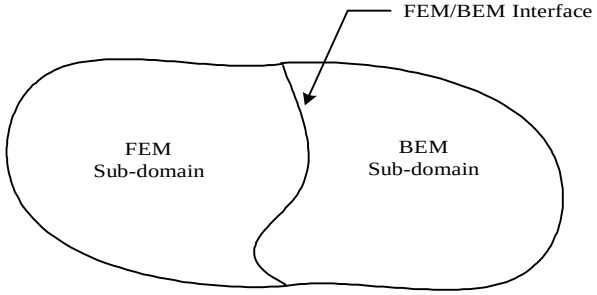

Figure 1: Domain Decomposed into FEM and BEM Sub-domains.

Equations (4) and (5) may be partitioned as follows:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} u_B^B \\ u_B^I \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{Bmatrix} q_B^B \\ q_B^I \end{Bmatrix} \tag{6}$$

and

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_F^F \\ u_F^I \end{Bmatrix} = \begin{Bmatrix} f_F^F \\ f_F^I \end{Bmatrix} \tag{7}$$

At the interface, the compatibility and equilibrium conditions should be satisfied, i.e.,

$$\{u_B^I\} = \{u_F^I\} \in \Gamma^I \tag{8}$$

$$\{f_F^I\} + [M]\{q_B^I\} = 0 \in \Gamma^I \tag{9}$$

where M is the converting matrix, which depends on the interpolation functions used to represent the flux on the interface.

In the remainder of this section we will summarize and review some of the existing domain decomposition coupling algorithms.

The sequential Dirichlet-Neumann coupling method may be described as follows [12,13]:

Set initial guess $\{u_{B,0}^I\} = \{\bar{u}\}$

Do for $n = 0, 1, 2, \ldots$

for the BEM sub-domain:

solve Equation (6) for $\{q_{B,n}^I\}$

for the FEM sub-domain:

solve Equation (9) for $\{f_F^I\}$

solve Equation (7) for $\{u_{F,n}^I\}$

apply $\{u_{B,n+1}^I\} = (1-\theta)\{u_{B,n}^I\} + \theta\{u_{F,n}^I\}$

where, θ is a relaxation parameter

until $\dfrac{\left\| \{u_{B,n+1}^I\} - \{u_{B,n}^I\} \right\|}{\left\| \{u_{B,n+1}^I\} \right\|} < \varepsilon$

where, ε is a given tolerance.

A drawback of this coupling algorithm is that it may not be suited for certain classes of problems where the natural boundary conditions are specified on the entire external boundary of the FEM sub-domain.

The parallel Neumann-Neumann coupling algorithm may be described as follows [11]:

Set initial guess $\{q_{B,0}^I\} = \{\bar{q}\}$ and $\{q_{F,0}^I\} = \{\bar{q}\}$

Do for $n = 0, 1, 2, \ldots$

for the BEM sub-domain:

solve Equation (6) for $\{u_{B,n}^I\}$

for the FEM sub-domain:

solve Equations (7) and (9) for $\{u_{F,n}^I\}$

apply $\{q_{B,n+1}^I\} = \{q_{B,n}^I\} + \beta(\{u_{F,n}^I\} - \{u_{B,n}^I\})$

apply $\{q_{F,n+1}^I\} = -\{q_{B,n+1}^I\}$

where, β is a relaxation parameter

until $\dfrac{\left\| \{u_{B,n+1}^I\} - \{u_{B,n}^I\} \right\|}{\left\| \{u_{B,n+1}^I\} \right\|} < \varepsilon$

where, ε is a given tolerance.

This algorithm may not be suited for problems where the natural boundary conditions are specified on the entire external boundary of the FEM or BEM sub-domains.

The parallel Dirichlet-Neumann coupling algorithm may be described as follows [11]:

Set initial guess $\{u_{B,0}^I\} = \{\bar{u}\}$ and $\{q_{F,0}^I\} = \{\bar{q}\}$

Do for $n = 0, 1, 2, \ldots$

for the BEM sub-domain:

solve Equation (6) for $\{q^I_{B,n}\}$

for the FEM sub-domain:

solve Equations (7) and (9) and get $\{u^I_{F,n}\}$

apply $\{u^I_{B,n+1}\}=(1-\gamma)\{u^I_{B,n}\}+\gamma\{u^I_{F,n}\}$

apply $\{q^I_{F,n+1}\}=-\{q^I_{B,n}\}$

where, γ is a relaxation parameter

until $\dfrac{\left\|\{u^I_{B,n+1}\}-\{u^I_{B,n}\}\right\|}{\left\|\{u^I_{B,n+1}\}\right\|}<\varepsilon$

where, ε is a given tolerance.

This coupling algorithm has the same drawbacks as that of the sequential Dirichlet-Neumann coupling method.

4. A New Interface Relaxation Coupling Algorithm

In this section, we utilize the interface relaxation algorithm, originally developed by Rice et al. [19] for the solution of elliptic differential equations and extended to higher dimensional analysis in Section 2, for the coupling of FEM and BEM. The interface relaxation-coupling algorithm may be described as follows:

Set initial guess $\{u^I_{F,0}\}=\{\bar{u}\}$ and $\{u^I_{B,0}\}=\{\bar{u}\}$

Do for $n=0,1,2,...$

for the BEM sub-domain

solve Equation (6) for $\{q^I_{B,n}\}$

for the FEM sub-domain:

solve $\{f^I_F\}=[M]\{q^I_F\}$ and Equation (7) for $\{q^I_{F,n}\}$

apply $\{u^I_{B,n+1}\}=\{u^I_{B,n}\}-\alpha(\{q^I_{B,n}\}+\{q^I_{F,n}\})$

apply $\{u^I_{F,n+1}\}=\{u^I_{B,n+1}\}$

where, α is a relaxation parameter

until $\dfrac{\left\|\{u^I_{B,n+1}\}-\{u^I_{B,n}\}\right\|}{\left\|\{u^I_{B,n+1}\}\right\|}<\varepsilon$

where, ε is a given tolerance.

5. Convergence of the Interface Relaxation Coupling Method

In this section we will investigate the convergence of the domain decomposition coupling method depicted in the previous section. We will show the conditions under which the iterations,

$$u^I_{B,n+1}=u^I_{B,n}-\alpha(q^I_{B,n}+q^I_{F,n}) \tag{10}$$

will converge to the true value of u^I.

Let us repartition Equations (4) and (5) as follows:

$$\begin{bmatrix}H_{11}&H_{12}&H_{13}\\H_{21}&H_{22}&H_{23}\\H_{31}&H_{32}&H_{33}\end{bmatrix}\begin{Bmatrix}u^K_B\\u^U_B\\u^I_B\end{Bmatrix}=\begin{bmatrix}G_{11}&G_{12}&G_{13}\\G_{21}&G_{22}&G_{23}\\G_{31}&G_{32}&G_{33}\end{bmatrix}\begin{Bmatrix}q^U_B\\q^K_B\\q^I_B\end{Bmatrix} \tag{11}$$

and

$$\begin{bmatrix}K_{11}&K_{12}&K_{13}\\K_{21}&K_{22}&K_{23}\\K_{31}&K_{32}&K_{33}\end{bmatrix}\begin{Bmatrix}u^K_F\\u^U_F\\u^I_F\end{Bmatrix}=\begin{Bmatrix}f^1_F\\f^2_F\\f^I_F\end{Bmatrix} \tag{12}$$

where,

u^K_B : non-interface known potentials in the BEM sub-domain

u^U_B : non-interface unknown potentials in the BEM sub-domain

u^K_F : non-interface known potentials in the FEM sub-domain

u^U_F : non-interface unknown potentials in the FEM sub-domain

q^K_B : non-interface known fluxes in the BEM sub-domain

q^U_B : non-interface unknown fluxes in the BEM sub-domain

After a series of matrix operations, Equation (10) may be written in the following form:

$$u^I_{B,n+1}=\left[(1-\alpha)I+\alpha C\right]u^I_{B,n}+\alpha c \tag{13}$$

where,

$$C=I-M^{-1}\left[K_{33}-K_{32}K_{22}^{-1}K_{23}\right]-\left[G_{33}-\begin{bmatrix}-G_{31}&H_{32}\end{bmatrix}\begin{bmatrix}-G_{11}&H_{12}\\-G_{21}&H_{22}\end{bmatrix}^{-1}\begin{bmatrix}G_{13}\\G_{23}\end{bmatrix}\right]^{-1}\left[H_{33}-\begin{bmatrix}-G_{31}&H_{32}\end{bmatrix}\begin{bmatrix}-G_{11}&H_{12}\\-G_{21}&H_{22}\end{bmatrix}^{-1}\begin{bmatrix}H_{13}\\H_{23}\end{bmatrix}\right]$$

and

$$c=\left[G_{33}-\begin{bmatrix}-G_{31}&H_{32}\end{bmatrix}\begin{bmatrix}-G_{11}&H_{12}\\-G_{21}&H_{22}\end{bmatrix}^{-1}\begin{bmatrix}G_{13}\\G_{23}\end{bmatrix}\right]^{-1}\left[\begin{bmatrix}-H_{31}&G_{32}\end{bmatrix}-\begin{bmatrix}-G_{31}&H_{32}\end{bmatrix}\begin{bmatrix}-G_{11}&H_{12}\\-G_{21}&H_{22}\end{bmatrix}^{-1}\begin{bmatrix}-H_{11}&G_{12}\\-H_{21}&G_{22}\end{bmatrix}\right]\begin{Bmatrix}u^K_B\\q^K_B\end{Bmatrix}+M^{-1}\left[K_{32}K_{22}^{-1}K_{21}-K_{31}\right]\{u^K_F\}-\left[K_{32}K_{22}^{-1}\right]\{f^2_F\}$$

Now, Equation (13) is an iterative method of the form:

$$X_{n+1}=D_\alpha X_n+d \tag{14}$$

which converges if and only if the set of eigenvalues $\sigma(D_\alpha)$ of the matrix D_α are contained in the unit ball $B(0,1)$ in the complex plane. Following the same procedures as in previous investigations [14,15], one may conclude that the coupling method will converge if:

$$x_i<1,\ i=1,2,.....N \tag{15}$$

and if we choose

$$\alpha < \min_{1 \le i \le N} \left\{ \frac{2(1-x_i)}{(1-x_i)^2 + y_i^2} \right\} \qquad (16)$$

where,

$\lambda_1 = x_1 + iy_1, ..., \lambda_N = x_N + iy_N$ are the eigenvalues of C.

The optimum value of the relaxation parameter ($\bar{\alpha}$) may be obtained as:

$$\bar{\alpha} = -\frac{Re(1^t(\lambda - 1))}{\|\lambda - 1\|^2} \qquad (17)$$

where, $\lambda^t = (\lambda_1 \quad \lambda_2 \ldots \lambda_n)$ and $1^t = (1 \quad 1 \ldots 1)$.

6. Numerical Example

Consider the case of the potential flow in a rectangular domain as shown in Figure 2. The two domains Ω_B and Ω_F are governed by Laplace equation. The rectangular domain is decomposed into the FEM and BEM sub-domains with $0 \le x \le a_F$ and $a_F \le x \le a_B$. The boundary conditions are selected such that $u(0, y) = 0$, $u(a, y) = 200$ and zero flux elsewhere. The problem is investigated for different values of a_B/a_F and K_B/K_F. For $a_B/a_F = 1$, the domain is modeled by 18 linear boundary elements and 40 linear triangular elements (see Figure 2). Tables 1 and 2 show the applicable range and the optimal values of the relaxation parameters determined experimentally using the different coupling algorithms described in Sections 3 and 4 and with different combinations of a_B/a_F and K_B/K_F. Beyond the values given by Table 1, the domain decomposition and the new interface relaxation coupling algorithms do not converge. The limit and optimum values of the parameter α are found in good agreement with those determined theoretically by Equations (15-17).

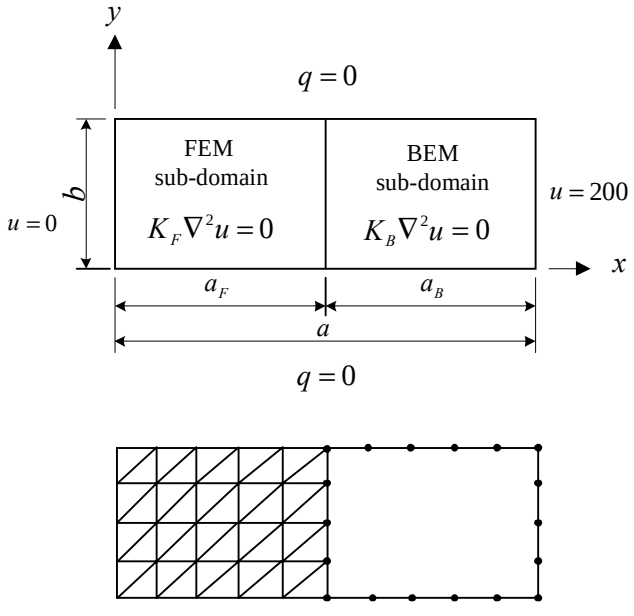

Figure 2: Potential Flow Problem and Discretization.

Table 1: Applicable range of the relaxation parameters.

a_B/a_F		K_B/K_F = 0.50	1.0	2.0
0.2	α	0.02-0.56	0.02-0.32	0.02-0.18
	β	0.02-0.98	0.02-0.98	0.02-0.98
	γ	0.02-0.36	0.02-0.18	0.02-0.08
	θ	0.02-0.56	0.02-0.32	0.02-0.18
1.0	α	0.02-1.32	0.02-0.98	0.02-0.66
	β	0.02-0.98	0.02-0.98	0.02-0.98
	γ	0.02-1.98	0.02-0.98	0.02-0.48
	θ	0.02-1.32	0.02-0.98	0.02-0.66
4.0	α	0.02-1.76	0.02-1.56	0.02-1.32
	β	0.02-0.98	0.02-0.98	0.02-0.98
	γ	0.02-2.28	0.02-2.66	0.02-1.98
	θ	0.02-1.76	0.02-1.56	0.02-1.32

Table 2: Optimal values of the relaxation parameters.

a_B/a_F		K_B/K_F = 0.50	1.0	2.0
0.2	$\bar{\alpha}$	0.28	0.16	0.10
	$\bar{\beta}$	0.24	0.30	0.38
	$\bar{\gamma}$	0.10	0.06	0.04
	$\bar{\theta}$	0.28	0.16	0.10
1.0	$\bar{\alpha}$	0.66	0.50	0.32
	$\bar{\beta}$	0.18	0.20	0.22
	$\bar{\gamma}$	0.38	0.18	0.12
	$\bar{\theta}$	0.66	0.50	0.32
4.0	$\bar{\alpha}$	0.88	0.78	0.66
	$\bar{\beta}$	0.10	0.14	0.18
	$\bar{\gamma}$	0.78	0.54	0.38
	$\bar{\theta}$	0.88	0.78	0.66

The same problem is reinvestigated with the boundary conditions shown in Figure 3. The problem is solved using the interface relaxation coupling algorithm and for $a_B/a_F = 1$ and $K_B/K_F = 1$. The results compare well with the theoretical solution and the range of the relaxation parameter is found to be 0.02-1.98 with an optimum value of 1. It should be noted that this simple problem is not solvable using the domain decomposition coupling methods presented in Section 3.

Conclusions

In this paper, we present a new interface relaxation algorithm for coupling the FEM and BEM. The method overcomes some of the limitations of the existing finite element and boundary element coupling methods. We further establish the general convergence conditions of the new coupling method.

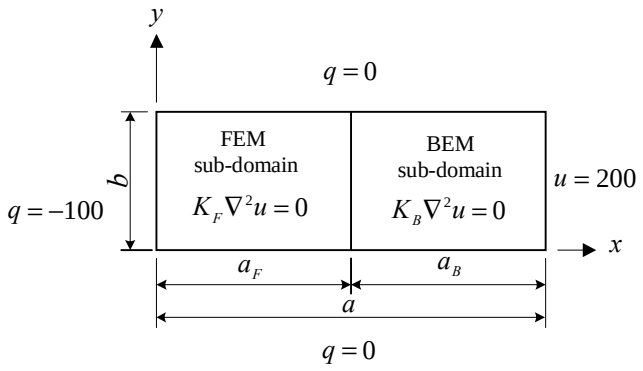

Figure 3: Potential Flow Problem with Neumann Boundary Conditions Specified on the Entire External Boundary of the FEM Sub-domain.

Acknowledgements

The authors gratefully acknowledge the support of Shinshu University and the Japan Society for the Promotion of Science, Japan.

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