Conditional stability in determining a periodic structure in a lossy medium and the Tikhonov regularization

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In this paper, we show conditional stability for an inverse problem of determining a periodic structure in diffractive optics from near field observations in a lossy medium, when we assume perfect reflection on the structure. Next we apply the conditional stability to obtain a convergence rate of regularized solutions by the Tikhonov regularization.

Key Words: uniqueness, conditional stability, inverse optics problems, Tikhonov regularization.

1 Formulation of the problem

We consider the scattering by the perfectly reflecting periodic structure in two dimensions. According to Bao(2), Bao et al.(3), Hettlich and Kirsch(9), Petit(12), we can formulate the problem as follows. Let $f \in C^2(R)$ be 2π-periodic, $f(x) < 0$ for $x \in R$ and on $\{(x,y); y = f(x),\, x \in R\}$, the perfect reflection condition is imposed. We set

$$\Omega_f = \{(x,y); y > f(x),\, x \in R\}. \qquad (1)$$

Then we regard $\partial\Omega_f = \{(x,y); y = f(x),\, x \in R\}$ as a periodic interface which we should determine by scattering data. For this, we introduce an incident field $u^I(x,y;k)$ given by

$$u^I(x,y;k) = \exp\{\mathrm{i}k(x\sin\theta - y\cos\theta)\}. \qquad (2)$$

Here $\mathrm{i} = \sqrt{-1}$, and $k = \ell_1 + \mathrm{i}\ell_2$ with $\ell_1, \ell_2 \in R$, is a wave number. Throughout this paper, we assume

$$0 < |\theta| < \frac{\pi}{2} \tag{3}$$

and

$$\ell_1^2 - \ell_2^2 \leq 0. \tag{4}$$

The condition (4) implies that the medium in Ω_f is lossy. Then the resulting scattering field $u^S(x, y; k)$ satisfies the Helmholtz equation with the perfect reflection boundary condition on $\partial\Omega_f$ and the boundedness condition at infinity:

$$\Delta u^S + k^2 u^S = 0 \quad \text{in} \quad \Omega_f. \tag{5}$$

$$u^S + u^I = 0 \quad \text{on} \quad \partial\Omega_f. \tag{6}$$

$$u^S \text{ is bounded as } y \to \infty. \tag{7}$$

Moreover, according to the form (2) of the incident wave, we pose the $(k \sin\theta)$-quasi-periodicity condition for u^S:

$$u^S(x + 2\pi, y; k) = \exp(2\pi \mathrm{i} k \sin\theta) u^S(x, y; k) \tag{8}$$

for all $(x, y) \in R^2$ (see e.g., Bao[(2)], Bao et al.[(3)], Hettlich and Kirsch[(9)]). For the unique existence of $u^S(f) = u^S(f)(x, y; k)$ satisfying (5) - (8), see Kirsch[(10)−(11)], Wilcox[(14)], for example. Then we will discuss

Inverse Problem of Diffractive Optics. Determine $y = f(x)$, $x \in R$ from the measurements $u^S(f)(x, 0; k)$, $x \in (0, 2\pi)$, where u^S satisfies (5) - (8).

For this inverse problem, from the mathematical point of view, the first issue is the uniqueness. That is, we should prove that the correspondence $f \leftrightarrow u^S(f)(x, 0; k)$ is one to one. For a lossy medium (i.e., $\mathrm{Im}k > 0$), see Bao[(2)], and for the case of $k \in R$, see Hettlich and Kirsch[(9)]. We further refer to Ammari[(1)] and Bruckner et al.[(5)] for other uniqueness results in our inverse problem.

Since real observation data are polluted with errors and in numerical computations, errors by discretization must be taken into consideration, the stability issue for our inverse problem is the next important theoretical subject. That is, we should clarify whether two periodic structures f and g are not far from each other, when the difference $u^S(f)(x, 0; k) - u^S(g)(x, 0; k)$ is small. In spite of the significance of the stability, there are very few papers on such a subject. Only Bao and Friedman[(4)] proved stability around a fixed f_0 which is of local character in the sense that f and g are restricted within a specially parametrized class. To the authors' knowledge, however, there are no works concerning the stability without such a specialized classs except for Bruckner et al.[(6)] where the medium is assumed to be non-lossy and k is not very big (i.e., $0 < k < \frac{1}{2\pi}$). The first purpose of this paper is to show similar stability in the lossy case (4).

We reformulate the problem. By the $(k\sin\theta)$-quasi-periodicity, setting

$$u(f) = u(f)(x,y;k)$$
$$= u^I(x,y;k) + u^S(f)(x,y;k), \tag{9}$$

we rewrite (5) - (8) in terms of the total field u:

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega_f \tag{10}$$

$$u = 0 \quad \text{on} \quad \partial\Omega_f \tag{11}$$

$$u(x+2\pi, y; k) = \exp(2\pi \mathrm{i} k \sin\theta) u(x,y;k). \tag{12}$$

$$u - u^I \text{ is bounded as } y \longrightarrow \infty. \tag{13}$$

Since k is so fixed that (4) is true, we simply write $u(f)(x,y)$ in place of $u(f)(x,y;k)$. Then our inverse problem is equivalent to: *determine $y = f(x)$, $x \in R$ from the measurements*

$$u(f)(x,0), \qquad x \in (0, 2\pi),$$

where u satisfies (10) - (13).

2 Conditional stability

For the stability, it is mathematically necessary to assume that unknown structures should satisfy some boundedness condition. Otherwise we can construct an example breaking the stability, as is suggested in Cheng et al.[7]. Also from a practical point of view, it is often reasonable to introduce some boundedness on the lengths, the curvatures, etc. of the unknown structure. Under suitable boundedness assumptions, we can restore the stability in our inverse problem which is called *conditional stability.*

In order to state our conditional stability, we need to define such boundedness for a set of unknowns. For fixed positive constants M_0, M, κ, and a_0, a such that $0 < M \le a_0 \le a$ and $0 < \kappa < 1$, we set

$$\begin{aligned} F = \{ & f \in C^{3+\kappa}(R); \|f\|_{C^{3+\kappa}[0,2\pi]} \le M_0, \\ & f \text{ is } (2\pi)\text{-periodic}, \\ & \frac{\mathrm{d}^j f}{\mathrm{d}x^j}(0) = \frac{\mathrm{d}^j f}{\mathrm{d}x^j}(2\pi), \quad j = 0,1,2,3, \\ & f(0) = f(2\pi) = -a_0, \\ & -a \le f(x) \le -M, \ 0 \le x \le 2\pi \} \end{aligned}$$

as an admissible set of unknown structures. Here and henceforth let

$$\|f\|_{C^{3+\kappa}[0,2\pi]} = \sum_{j=0}^{3} \max_{0 \le x \le 2\pi} \left| \frac{\mathrm{d}^j f}{\mathrm{d}x^j}(x) \right| + \sup_{0 \le x, x' \le 2\pi, x \ne x'} \frac{\left| \frac{\mathrm{d}^3 f}{\mathrm{d}x^3}(x) - \frac{\mathrm{d}^3 f}{\mathrm{d}x^3}(x') \right|}{|x - x'|^\kappa}.$$

In other words, F has a uniform bound in $C^{3+\kappa}[0,2\pi]$. We recall

$$\Omega_f = \{(x,y); y > f(x), \quad x \in R\}$$

for $f \in F$.

For $f_j \in F$, $j = 1,2$, let us consider

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega_{f_j}$$

$$u = 0 \quad \text{on} \quad \partial\Omega_{f_j}$$

and let us assume that

$$u \text{ is } (k\sin\theta)\text{-quasi-periodic, i.e.,}$$
$$u(x+2\pi, y) = \exp(2\pi \mathrm{i} k \sin\theta) u(x,y)$$

and

$$u - u^I \text{ is bounded as } y \to \infty.$$

We are ready to state our main result on the conditional stability in determining $f_1, f_2 \in F$:

Theorem 1. *We assume (4). Then there exists a constant $C = C(k, \theta, F) > 0$ such that*

$$\max_{0\le x\le 2\pi} |f_1(x) - f_2(x)|$$
$$\le \frac{C}{\left|\log\left|\log \frac{1}{\|(u(f_1)-u(f_2))(\cdot,0)\|_{H^1(0,2\pi)}}\right|\right|}$$

provided that $f_1, f_2 \in F$.

Here and henceforth we set

$$\|(u(f_1) - u(f_2))(\cdot, 0)\|_{H^1(0,2\pi)}$$
$$= \left(\int_0^{2\pi} \left\{ |(u(f_1) - u(f_2))(x,0)|^2 \right.\right.$$
$$+ \left.\left. \left|\frac{\partial}{\partial x}(u(f_1) - u(f_2))(x,0)\right|^2 \right\} \mathrm{d}x \right)^{\frac{1}{2}}.$$

Our conditional stability is doubly logarithmic and is rather weak. However, this kind of weak conditional stability is quite common in determining interfaces (e.g., Cheng et al.[7], Rondi[13]), which reflects the severe ill-posedness (i.e., very strong instability) in our inverse problem. It is extremely difficult to improve the conditional stability.

The proof of Theorem 1 is very technical and can be carried out very similarly to Bruckner et al.[6] except that we have to apply the maximum principle for the Helmholtz equation: let $k \in C$ satisfy (4) and let $u \in C^2(D) \cap C(\overline{D})$ satisfy $\Delta u + k^2 u = 0$ in D, where $D \subset R^2$ is a bounded domain. Then

$$\max_{(x,y)\in\overline{D}} |u(x,y)| = \max_{(x,y)\in\partial D} |u(x,y)|. \qquad (14)$$

Here we note that u is complex-valued. For the completeness, we will prove (14) in Appendix.

3 Tikhonov regularization

The conditional stability is very helpful for guaranteeing convergence rates of Tikhonov's regularized solutions and, on the basis of Theorem 1, we apply Cheng and Yamamoto[8] to establish a convergence rate of the Tikhonov regularized solutions with an adequate choice of regularizing parameters.

Let us consider the following functional which contains a positive parameter α:

$$G(f) = \|u(f)(\cdot,0) - u^\delta\|^2_{L^2(0,2\pi)}$$
$$+\alpha\|f\|^2_{H^4(0,2\pi)}$$

where u^δ is the measured data which contains some error. We assume to know its error bound. That is, for the exact solution $u(f_0)(\cdot, 0)$, let us assume that

$$\|u(f_0)(\cdot,0) - u^\delta\|_{L^2(0,2\pi)} < \delta,$$

where $\delta > 0$ is an *a priori* error bound.

Here and henceforth, we set

$$\|f\|_{H^4(0,2\pi)} = \left(\sum_{j=0}^{4} \left\| \frac{d^j f}{dx^j} \right\|^2_{L^2(0,2\pi)} \right)^{\frac{1}{2}},$$

$$\|f\|_{L^2(0,2\pi)} = \left(\int_0^{2\pi} |f(x)|^2 dx \right)^{\frac{1}{2}}$$

and $H^4(0,2\pi) = \{f; \|f\|_{H^4(0,2\pi)} < \infty\}$.

We suppose that the exact solution f_0 to the inverse problem is smooth, that is, $f_0 \in H^4(0,2\pi)$. As approximations to f_0, we take a quasi-minimizer of the functional G. Then, in order to obtain a reasonable convergence rate of the approximations to f_0, an *a priori* choice strategy for α is essential. Combining Theorem 1 with Cheng and Yamamoto[8], we can readily prove

Theorem 2. *Suppose that $\alpha = c\delta$ and $f_\alpha^\delta \in H^4(0,2\pi)$ satisfies*

$$G(f_\alpha^\delta) \le \inf_{f \in H^4(0,2\pi)} G(f) + \delta^2.$$

Here $c > 0$ is a constant which is independent of δ. Then we have

$$\max_{0 \le x \le 2\pi} |f_\alpha^\delta(x) - f_0(x)| \le \frac{C}{\left|\log\left|\log \frac{1}{\delta}\right|\right|}$$

where $C > 0$ is a positive constant which depends on f_0.

4 Conclusions

(1) In the case of lossy media, we show a conditional stability result for the inverse optics problem and the stability rate is doubly logarithmic.

(2) With an adequate choice of the Tikhonov regularizing parameters, we can gain the convergence of the regularized solutions towards the exact solution and the convergence rate is same as in the conditional stability. The choice of α should be proportional to δ, the noise level.

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Appendix. Proof of (14). We set $k^2 = \mu_1 + \mathrm{i}\mu_2$ with $\mu_1, \mu_2 \in R$ and $u = v + \mathrm{i}w$ with real-valued functions v, w. Then from the Helmholtz equation, we have $\Delta v + \mu_1 v - \mu_2 w = 0$ and $\Delta w + \mu_1 w + \mu_2 v = 0$ in D. We set $W = v^2 + w^2$. Since (4) implies that $\mu_1 \le 0$, we can drectly see that

$$\begin{aligned} \Delta W &= 2(|\nabla v|^2 + |\nabla w|^2) + 2v\Delta v + 2w\Delta w \\ &\ge 2v(-\mu_1 v + \mu_2 w) + 2w(-\mu_1 w - \mu_2 v) \\ &\ge -2\mu_1 W \ge 0 \end{aligned}$$

in D. Consequently the maximum principle implies that $\sup_D W = \sup_{\partial D} W$. Thus the proof of (14) is complete.

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