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BEM-BEM COUPLING AND FEM-BEM COUPLING VIA INTERFACE RELAXATION

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This paper reviews existing domain decomposition finite element-boundary element coupling algorithms. It further adds a number of interface relaxation algorithms for boundary element-boundary element coupling and finite element-boundary element coupling. These coupling algorithms overcome some of the limitations of the existing ones. Numerical examples are also given.

Keywords: Boundary element method; Finite element method; Interface relaxation; Coupling.

1. Introduction

The idea of coupling the finite element method (FEM) and the boundary element method (BEM) is by now well known as an effective analysis tool, which makes use of their individual merits. The first proposed FEM and BEM coupled formulation was presented by Zienkiewicz et al. [1]. An extensive literature survey on this topic can be found in [2-5]. The conventional coupling methods employ an entire unified equation for the whole domain by altering the formulation of one of the methods to make it compatible with the other. However, the implementation of the conventional coupling procedures requires a suitably integrated FEM-BEM software environment, which necessitates merging two different kinds of programs. In order to preserve the nature of both the FEM and BEM, the domain decomposition FEM-BEM coupling methods have been developed [6-10]. In these methods, separate computing for the BEM and FEM sub-domains and successive renewal of the degrees of freedom on the interface of both sub-domains are performed to reach the final convergence. The domain decomposition coupling methods appear to be promising. However, the important issue of convergence of the domain decomposition coupling methods is not fully addressed. Moreover, some existing domain decomposition coupling methods are not applicable for solving problems at which the Neumann boundary conditions are specified on the entire external boundary of the FEM or BEM sub-domains.

In boundary element analysis, sub-domain partition may be employed when the domains under consideration are governed by individual differential equations and/or constructed of different materials. Besides, in the case of domain with complicated boundary profile, the domain may be decomposed for better computational efficiency and accuracy. In the conventional BEM-BEM coupling formulation, sub-domains equations are collected into a system of equations while satisfying the continuity and equilibrium conditions along the interfaces. However, the conventional techniques requires access to the system matrices which will limit the BEM-BEM coupling to users having access to source codes. Kamiya et al. [11] utilized the Neumann-Neumann and Dirichlet-Neumann renewal algorithms for BEM-BEM coupling.

Interface relaxation is more general than the traditional domain decomposition methods in that it allows unrelated PDE problems on different sub-domains, see references [12-16]. Elleithy and Tanaka [17] presented two interface relaxation algorithms for FEM-BEM coupling.

The main objective of this paper is to formulate a population of FEM-BEM and BEM-BEM coupling algorithms. Several interface relaxation coupling algorithms are presented. The algorithms make it easy to handle different physical models and to reuse existing FEM and BEM codes/software.

2. FEM-BEM Coupling

Consider Figure 1, where the domain of the original problem is decomposed into FEM and BEM sub-domains. The corresponding boundary integral equation for the BEM sub-domain is given by:

$$[H]\{u\}=[G]\{q\} \quad on \quad \Gamma_B \tag{1}$$

where, $\{u\}$ and $\{q\}$ are column matrices containing the boundary nodal values for the potentials (displacements) and the fluxes (tractions). $[H]$ and $[G]$ are influence coefficient matrices. For the FEM sub-domain, the assembled element equations are given by:

$$[K]\{u\}=\{f\} \quad on \quad \Omega_F \tag{2}$$

where $[K]$ is the stiffness matrix for the system, and $\{u\}$ and $\{f\}$ are the nodal potentials (displacements) and integrated flux (force) vectors, respectively. Now, let us define the following vectors:

$\{u_B^I\}$: interface potentials (displacements), approached from the BEM sub-domain

$\{u_B^B\}$: non-interface potentials (displacements) in the BEM sub-domain

$\{u_F^I\}$: interface potentials (displacements), approached from the FEM sub-domain

$\{u_F^F\}$: non-interface potentials (displacements) in the FEM sub-domain.

Similarly, one can define the flux (tractions) and the integrated flux (force) vectors for the BEM and FEM sub-domains, respectively.

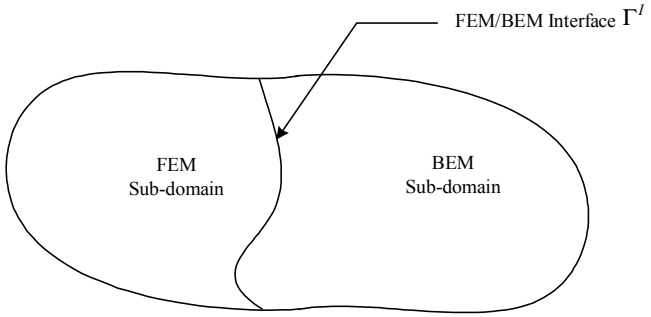


Figure 1: Domain decomposed into FEM and BEM sub-domains.

Equations (1) and (2) may be partitioned as follows:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} u_B^B \\ u_B^I \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{Bmatrix} q_B^B \\ q_B^I \end{Bmatrix} \tag{3}$$

and

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_F^F \\ u_F^I \end{Bmatrix} = \begin{Bmatrix} f_F^F \\ f_F^I \end{Bmatrix} \tag{4}$$

At the interface, the compatibility and equilibrium conditions should be satisfied, i.e.,

$$\{ u_B^I \} = \{ u_F^I \} \quad on \ \Gamma^I \tag{5}$$

$$\{ q_B^I \} = -\{ q_F^I \} \quad on \ \Gamma^I \tag{6}$$

The relationship between $\{ q_F^I \}$ and $\{ f_F^I \}$ may be given as:

$$\{ f_F^I \} = [M] \{ q_F^I \} \quad on \ \Gamma^I \tag{7}$$

where $[M]$ is the converting matrix, which depends on the interpolation functions used to represent the flux (tractions) on the interface. Using Equations (6) and (7) the equilibrium conditions at the interface may be written as:

$$\{ f_F^I \} + [M] \{ q_B^I \} = 0 \quad on \ \Gamma^I \tag{8}$$

In Section 2.1, we briefly review the existing domain decomposition FEM-BEM coupling algorithms. Sections 2.2 and 2.3 summarize the interface relaxation FEM-BEM coupling algorithms presented by Elleithy and Tanaka [17]. Section 2.4 presents a new interface relaxation algorithm for the coupling of FEM and BEM.

2.1. Domain decomposition coupling algorithms

The sequential Dirichlet-Neumann FEM-BEM coupling algorithm may be described as follows [9,10]:

Set the initial guess $\{ u_{B,0}^I \}$.

Do for $n = 0, 1, 2, \ldots$, until convergence

for the BEM sub-domain, solve:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} u_{B,n}^B \\ u_{B,n}^I \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{Bmatrix} q_{B,n}^B \\ q_{B,n}^I \end{Bmatrix} \text{ for } \{ q_{B,n}^I \}$$

for the FEM sub-domain, solve:

$$\{ f_{F,n}^I \} + [M] \{ q_{B,n}^I \} = 0 \text{ for } \{ f_{F,n}^I \}$$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_{F,n}^F \\ u_{F,n}^I \end{Bmatrix} = \begin{Bmatrix} f_{F,n}^F \\ f_{F,n}^I \end{Bmatrix} \text{ for } \{ u_{F,n}^I \}$$

apply $\{ u_{B,n+1}^I \} = (1-\theta) \{ u_{B,n}^I \} + \theta \{ u_{F,n}^I \}$

where θ is a relaxation parameter to ensure and/or accelerate convergence.

The parallel Neumann-Neumann FEM-BEM coupling algorithm may be described as follows [8]:

Set the initial guess $\{ q_{B,0}^I \}$ and $\{ q_{F,0}^I \}$.

Do for $n = 0, 1, 2, \ldots$, until convergence

for the BEM sub-domain, solve:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} u_{B,n}^B \\ u_{B,n}^I \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{Bmatrix} q_{B,n}^B \\ q_{B,n}^I \end{Bmatrix} \text{ for } \{ u_{B,n}^I \}$$

for the FEM sub-domain, solve:

$$\{ f_{F,n}^I \} = [M] \{ q_{F,n}^I \} \text{ for } \{ f_{F,n}^I \}$$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_{F,n}^F \\ u_{F,n}^I \end{Bmatrix} = \begin{Bmatrix} f_{F,n}^F \\ f_{F,n}^I \end{Bmatrix} \text{ for } \{ u_{F,n}^I \}$$

apply $\{ q_{B,n+1}^I \} = \{ q_{B,n}^I \} + \beta \left(\{ u_{F,n}^I \} - \{ u_{B,n}^I \} \right)$

apply $\{ q_{F,n+1}^I \} = -\{ q_{B,n+1}^I \}$

where β is a relaxation parameter to ensure and/or accelerate convergence.

The parallel Dirichlet-Neumann FEM-BEM coupling algorithm may be described as follows [8]:

Set initial guess $\{ u_{B,0}^I \}$ and $\{ q_{F,0}^I \}$.

Do for $n = 0, 1, 2, \ldots$, until convergence

for the BEM sub-domain, solve:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} u_{B,n}^B \\ u_{B,n}^I \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{Bmatrix} q_{B,n}^B \\ q_{B,n}^I \end{Bmatrix} \text{ for } \{ q_{B,n}^I \}$$

for the FEM sub-domain, solve:

$$\{ f_{F,n}^I \} = [M] \{ q_{F,n}^I \} \text{ for } \{ f_{F,n}^I \}$$

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u_{F,n}^F \\ u_{F,n}^I \end{Bmatrix} = \begin{Bmatrix} f_{F,n}^F \\ f_{F,n}^I \end{Bmatrix} \text{ for } \{ u_{F,n}^I \}$$

apply $\{ u_{B,n+1}^I \} = (1-\gamma) \{ u_{B,n}^I \} + \gamma \{ u_{F,n}^I \}$

apply $\{ q_{F,n+1}^I \} = -\{ q_{B,n}^I \}$

where γ is a relaxation parameter to ensure and/or accelerate convergence.

A drawback of the existing domain decomposition FEM-BEM coupling algorithms is that they produce non-unique solutions for problems at which the Neumann boundary conditions are specified on the entire external boundary of the FEM sub-domain.

2.2. Geometric contraction based FEM-BEM coupling algorithm

Rice et al. [16] presented an interface relaxation algorithm for the solution of elliptic differential equations. The algorithm estimates a new solution for each sub-domain by solving a Dirichlet problem. Elleithy and Tanaka [17] utilized the algorithm for FEM-BEM coupling. The coupling algorithm may be described as follows:

Set initial guess $\{ u_{F,0}^I \}$ and $\{ u_{B,0}^I \}$.

For $n = 0, 1, 2, \ldots$, do until convergence

for the BEM sub-domain, solve:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} u^B_{B,n} \\ u^I_{B,n} \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{Bmatrix} q^B_{B,n} \\ q^I_{B,n} \end{Bmatrix} \text{ for } \left\{ q^I_{B,n} \right\}$$

for the FEM sub-domain, solve:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u^F_{F,n} \\ u^I_{F,n} \end{Bmatrix} = \begin{Bmatrix} f^F_{F,n} \\ f^I_{F,n} \end{Bmatrix} \text{ and}$$

$$\left\{ f^I_{F,n} \right\} = [M] \left\{ q^I_{F,n} \right\} \text{ for } \left\{ q^I_{F,n} \right\}$$

apply $\left\{ u^I_{B,\,n+1} \right\} = \left\{ u^I_{B,\,n} \right\} - \alpha \left(\left\{ q^I_{B,n} \right\} + \left\{ q^I_{F,n} \right\} \right)$

apply $\left\{ u^I_{F,\,n+1} \right\} = \left\{ u^I_{B,\,n+1} \right\}$

where α is a relaxation parameter to ensure and/or accelerate convergence.

The geometric contraction based FEM-BEM coupling algorithm is suited for solving problems at which the Neumann boundary conditions are specified on the entire external boundary of the FEM or BEM sub-domains.

2.3. Robin relaxation FEM-BEM coupling algorithm

This method predicts new values at the interface by making a convex combination of Dirichlet and Neumann data from the neighboring sub-domains [14]. Elleithy and Tanaka [17] utilized the algorithm for FEM-BEM coupling. The coupling algorithm may be described as follows:

Define $\left\{ g^I_{B,\,n+1} \right\} = -\left\{ q^I_{F,\,n} \right\} + \rho \left\{ u^I_{F,\,n} \right\}$ and

$\left\{ g^I_{F,\,n+1} \right\} = -\left\{ q^I_{B,\,n} \right\} + \rho \left\{ u^I_{B,\,n} \right\}$ where ρ is a relaxation parameter to ensure and/or accelerate convergence.

Set initial guess $\left\{ u^I_{F,0} \right\}$ and $\left\{ u^I_{B,0} \right\}$.

For $n = 0, 1, 2, \ldots$, do until convergence

for the BEM sub-domain and with the Robin boundary conditions on the interface, solve:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} u^B_{B,n} \\ u^I_{B,n} \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{Bmatrix} q^B_{B,n} \\ q^I_{B,n} \end{Bmatrix} \text{ for } \left\{ u^I_{B,n} \right\}$$

and $\left\{ q^I_{B,n} \right\}$

for the FEM sub-domain and with the Robin boundary conditions on the interface, solve:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u^F_{F,n} \\ u^I_{F,n} \end{Bmatrix} = \begin{Bmatrix} f^F_{F,n} \\ f^I_{F,n} \end{Bmatrix} \text{ and}$$

$$\left\{ f^I_{F,n} \right\} = [M] \left\{ q^I_{F,n} \right\} \text{ for } \left\{ u^I_{F,n} \right\} \text{ and } \left\{ q^I_{F,n} \right\}$$

apply $\left\{ g^I_{B,\,n+1} \right\} = -\left\{ q^I_{F,\,n} \right\} + \rho \left\{ u^I_{F,\,n} \right\}$

apply $\left\{ g^I_{F,\,n+1} \right\} = -\left\{ q^I_{B,\,n} \right\} + \rho \left\{ u^I_{B,\,n} \right\}$

This coupling algorithm is also capable of handling problems at which the Neumann boundary conditions are specified on the entire external boundary of the FEM or BEM sub-domains.

2.4. Dirichlet/Neumann averaging FEM-BEM coupling algorithm

The Dirichlet/Neumann averaging is one of the simplest algorithms for the solution of composite PDEs [16]. In this section, we utilize the interface relaxation algorithm for FEM-BEM coupling. The coupling algorithm may be described as follows:

Set initial guess $\left\{ u^I_{F,0} \right\}$ and $\left\{ u^I_{B,0} \right\}$.

For $n = 0, 2, 4, \ldots$, do until convergence

for the BEM sub-domain, solve:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} u^B_{B,n} \\ u^I_{B,n} \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{Bmatrix} q^B_{B,n} \\ q^I_{B,n} \end{Bmatrix} \text{ for } \left\{ q^I_{B,n} \right\}$$

for the FEM sub-domain, solve:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u^F_{F,n} \\ u^I_{F,n} \end{Bmatrix} = \begin{Bmatrix} f^F_{F,n} \\ f^I_{F,n} \end{Bmatrix} \text{ and}$$

$$\left\{ f^I_{F,n} \right\} = [M] \left\{ q^I_{F,n} \right\} \text{ for } \left\{ q^I_{F,n} \right\}$$

apply $\left\{ q^I_{B,\,n+1} \right\} = (1 - \varphi_1) \left\{ q^I_{B,n} \right\} - \varphi_1 \left\{ q^I_{F,\,n} \right\}$

apply $\left\{ q^I_{F,\,n+1} \right\} = (1 - \varphi_1) \left\{ q^I_{F,n} \right\} - \varphi_1 \left\{ q^I_{B,\,n} \right\}$

for the BEM sub-domain, solve:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} u^B_{B,n+1} \\ u^I_{B,n+1} \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{Bmatrix} q^B_{B,n+1} \\ q^I_{B,n+1} \end{Bmatrix} \text{ for } \left\{ u^I_{B,n+1} \right\}$$

for the FEM sub-domain, solve:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} u^F_{F,n+1} \\ u^I_{F,n+1} \end{Bmatrix} = \begin{Bmatrix} f^F_{F,n+1} \\ f^I_{F,n+1} \end{Bmatrix} \text{ and}$$

$$\left\{ f^I_{F,n+1} \right\} = [M] \left\{ q^I_{F,n+1} \right\} \text{ for } \left\{ u^I_{F,n+1} \right\}$$

apply $\left\{ u^I_{B,\,n+2} \right\} = (1 - \varphi_2) \left\{ u^I_{B,n+1} \right\} + \varphi_2 \left\{ u^I_{F,\,n+1} \right\}$

apply $\left\{ u^I_{F,\,n+2} \right\} = (1 - \varphi_2) \left\{ u^I_{F,n+1} \right\} + \varphi_2 \left\{ u^I_{B,\,n+1} \right\}$

where φ_1 and φ_2 are relaxation parameters to ensure and/or accelerate convergence.

3. BEM-BEM coupling

Consider Figure 2, where the domain of the original problem is decomposed into m BEM sub-domains, $\Omega_1, \Omega_2, \ldots, \Omega_m$. Let $S(i)$ be the set of indices of those sub-domains that are neighbors of the sub-domain Ω_i. Now, let us define the following vectors:

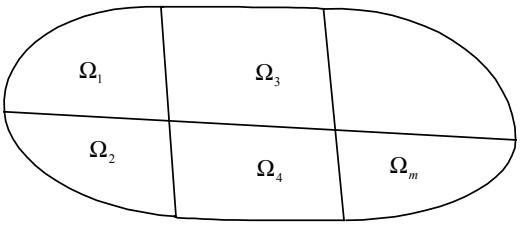
u^I_i: potentials (displacements) in the sub-domain Ω_i at the interface with all neighboring sub-domains, i.e., $u^I_i = \bigcup u^{I(ij)}_i$ for all $j \in S(i)$.

u^B_i: non-interface potentials (displacements) in the sub-domain Ω_i.

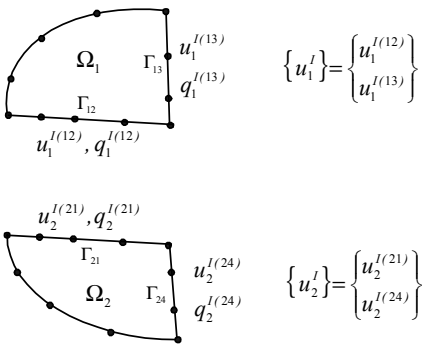
Similarly, one can define the flux (tractions) vectors for the BEM sub-domains. The discretized integral equations for the BEM sub-domains $(i = 1, \ldots, m)$ may be written in the following form:

$$\begin{bmatrix} H_{i,11} & H_{i,12} \\ H_{i,21} & H_{i,22} \end{bmatrix} \begin{Bmatrix} u^B_i \\ u^I_i \end{Bmatrix} = \begin{bmatrix} G_{i,11} & G_{i,12} \\ G_{i,21} & G_{i,22} \end{bmatrix} \begin{Bmatrix} q^B_i \\ q^I_i \end{Bmatrix} \tag{9}$$

At the interface (Γ_{kl}) between two adjacent sub-domains, Ω_k and Ω_l, the compatibility and equilibrium conditions should be satisfied, i.e.,



(a) Domain of the original problem decomposed into *m* subdomains



(b) BEM modeling for Ω_1 and Ω_2

Figure 2: BEM-BEM coupling. (a) Domain decomposed into m sub-domains. (b) BEM modeling.

$$\left\{ u_k^{I(kl)} \right\} = \left\{ u_l^{I(lk)} \right\} \tag{10}$$

$$\left\{ q_k^{I(kl)} \right\} = -\left\{ q_l^{I(lk)} \right\} \tag{11}$$

In Section 3.1, we briefly review the Neumann-Neumann BEM-BEM domain decomposition coupling algorithm [14]. Sections 3.2 through 3.4 present several new interface relaxation BEM-BEM coupling algorithms.

3.1. Neumann-Neumann BEM-BEM coupling algorithm

The coupling algorithm may be described as follows [11]:

Set initial guess $\left\{ q_{i,0}^I \right\}$ for the BEM sub-domains $(i = 1, \ldots, m)$.

Do for $n = 0, 1, 2, \ldots$ until convergence

for $i = 1, \ldots, m$

solve $\begin{bmatrix} H_{i,11} & H_{i,12} \\ H_{i,21} & H_{i,22} \end{bmatrix} \begin{Bmatrix} u_{i,n}^B \\ u_{i,n}^I \end{Bmatrix} = \begin{bmatrix} G_{i,11} & G_{i,12} \\ G_{i,21} & G_{i,22} \end{bmatrix} \begin{Bmatrix} q_{i,n}^B \\ q_{i,n}^I \end{Bmatrix}$ for $\left\{ u_{i,n}^I \right\}$

apply $\left\{ q_{i,\,n+1}^{I(ij)} \right\} = \left\{ q_{i,\,n}^{I(ij)} \right\} + \beta \left(\left\{ u_{j,n}^{I(ji)} \right\} - \left\{ u_{i,n}^{I(ij)} \right\} \right)$ for all $j \in S(i)$

where, β is a relaxation parameter to ensure and/or accelerate convergence.

3.2. Geometric contraction based BEM-BEM coupling algorithm

The coupling algorithm may be described as follows:

Set initial guess $\left\{ u_{i,0}^I \right\}$ for the BEM sub-domains $(i = 1, \ldots, m)$.

For $n = 0, 1, 2, \ldots$, do until convergence

for $i = 1, \ldots, m$

solve $\begin{bmatrix} H_{i,11} & H_{i,12} \\ H_{i,21} & H_{i,22} \end{bmatrix} \begin{Bmatrix} u_{i,n}^B \\ u_{i,n}^I \end{Bmatrix} = \begin{bmatrix} G_{i,11} & G_{i,12} \\ G_{i,21} & G_{i,22} \end{bmatrix} \begin{Bmatrix} q_{i,n}^B \\ q_{i,n}^I \end{Bmatrix}$ for $\left\{ q_{i,n}^I \right\}$

apply $\left\{ u_{i,\,n+1}^{I(ij)} \right\} = \left\{ u_{i,\,n}^{I(ij)} \right\} - \alpha \left(\left\{ q_{i,n}^{I(ij)} \right\} + \left\{ q_{j,n}^{I(ji)} \right\} \right)$ for all $j \in S(i)$

where α is a relaxation parameter to ensure and/or accelerate convergence.

3.3. Robin relaxation BEM-BEM coupling algorithm

This coupling algorithm may be described as follows:

Define $\left\{ g_{i,\,n+1}^{I(ij)} \right\} = -\left\{ q_{j,\,n}^{I(ji)} \right\} + \rho \left\{ u_{j,\,n}^{I(ji)} \right\}$ for all $j \in S(i)$ where ρ is a relaxation parameter to ensure and/or accelerate convergence.

Set initial guess $\left\{ u_{i,0}^I \right\}$ for the BEM sub-domains $(i = 1, \ldots, m)$.

For $n = 0, 1, 2, \ldots$, do until convergence

for $i = 1, \ldots, m$

with the Robin boundary conditions on the interface

solve $\begin{bmatrix} H_{i,11} & H_{i,12} \\ H_{i,21} & H_{i,22} \end{bmatrix} \begin{Bmatrix} u_{i,n}^B \\ u_{i,n}^I \end{Bmatrix} = \begin{bmatrix} G_{i,11} & G_{i,12} \\ G_{i,21} & G_{i,22} \end{bmatrix} \begin{Bmatrix} q_{i,n}^B \\ q_{i,n}^I \end{Bmatrix}$ for $\left\{ u_{i,n}^I \right\}$ and $\left\{ q_{i,n}^I \right\}$

apply $\left\{ g_{i,\,n+1}^{I(ij)} \right\} = -\left\{ q_{j,\,n}^{I(ji)} \right\} + \rho \left\{ u_{j,\,n}^{I(ji)} \right\}$ for all $j \in S(i)$

3.4. Dirichlet/Neumann averaging BEM-BEM coupling algorithm

The coupling algorithm may be described as follows:

Set initial guess $\left\{ u_{i,0}^I \right\}$ for the BEM sub-domains $(i = 1, \ldots, m)$.

For $n = 0, 2, 4, \ldots$, do until convergence

for $i = 1, \ldots, m$

solve $\begin{bmatrix} H_{i,11} & H_{i,12} \\ H_{i,21} & H_{i,22} \end{bmatrix} \begin{Bmatrix} u_{i,n}^B \\ u_{i,n}^I \end{Bmatrix} = \begin{bmatrix} G_{i,11} & G_{i,12} \\ G_{i,21} & G_{i,22} \end{bmatrix} \begin{Bmatrix} q_{i,n}^B \\ q_{i,n}^I \end{Bmatrix}$ for $\left\{ q_{i,n}^I \right\}$

apply $\left\{ q_{i,\,n+1}^{I(ij)} \right\} = (1 - \varphi_1) \left\{ q_{i,\,n}^{I(ij)} \right\} - \varphi_1 \left\{ q_{j,n}^{I(ji)} \right\}$ for all $j \in S(i)$

solve $\begin{bmatrix} H_{i,11} & H_{i,12} \\ H_{i,21} & H_{i,22} \end{bmatrix} \begin{Bmatrix} u_{i,n+1}^B \\ u_{i,n+1}^I \end{Bmatrix} = \begin{bmatrix} G_{i,11} & G_{i,12} \\ G_{i,21} & G_{i,22} \end{bmatrix} \begin{Bmatrix} q_{i,n+1}^B \\ q_{i,n+1}^I \end{Bmatrix}$ for $\left\{ u_{i,n+1}^I \right\}$

apply $\left\{ u_{i,\,n+2}^{I(ij)} \right\} = (1 - \varphi_2) \left\{ u_{i,\,n+1}^{I(ij)} \right\} + \varphi_2 \left\{ u_{j,n+1}^{I(ji)} \right\}$ for all $j \in S(i)$

where φ_1 and φ_2 are relaxation parameters to ensure and/or accelerate convergence.

4. Numerical Examples

As an FEM-BEM coupling example, consider the potential flow problem shown in Figure 3(a). The domain of the original problem is decomposed into Ω_B and Ω_F which are governed by Laplace equation. The boundary conditions are selected such that $u(0,y)=0$, $u(a,y)=200$ and zero flux elsewhere. Figure 3(b) shows the discretization for $a_B/a_F=1$, where the domain is modeled by 18 linear boundary elements and 40 linear triangular finite elements. For $a_B/a_F=1$ and $K_B/K_F=2$, Figure 4 shows the applicable range and optimal values of the relaxation parameters obtained using different interface relaxation FEM-BEM coupling algorithms. Note that for the Dirichlet/Neumann averaging algorithm we set $\varphi_1=\varphi_2=\varphi$. For the Robin interface relaxation FEM-BEM coupling algorithm the relaxation parameter might be assigned the value $\rho=1.0$, as this is determined as the optimal value. With the increase of ρ , the algorithm remains converged but with a larger number of iterations.

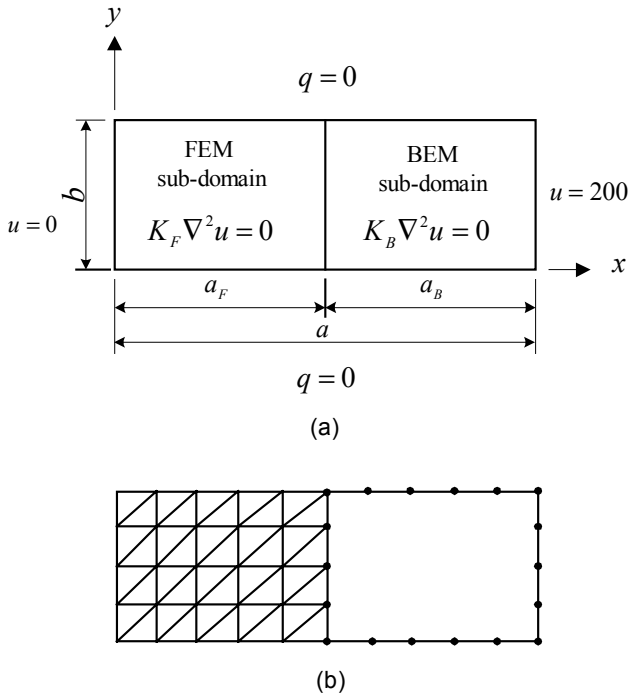


Figure 3: Potential flow problem. (a) Domain decomposed into FEM and BEM sub-domains. (b) FEM-BEM discretization.

El-Gebeily et al. established the convergence conditions for the sequential Dirichlet-Neumann, parallel Dirichlet-Neumann and parallel Neumann-Neumann FEM-BEM domain decomposition coupling algorithms [18]. We verified the applicable ranges and optimal values of β , γ and θ , using the convergence and optimal convergence conditions given in reference [18]. For the Robin interface relaxation and the geometric contraction based coupling algorithms, the convergence conditions are to be established and this may be considered for future work.

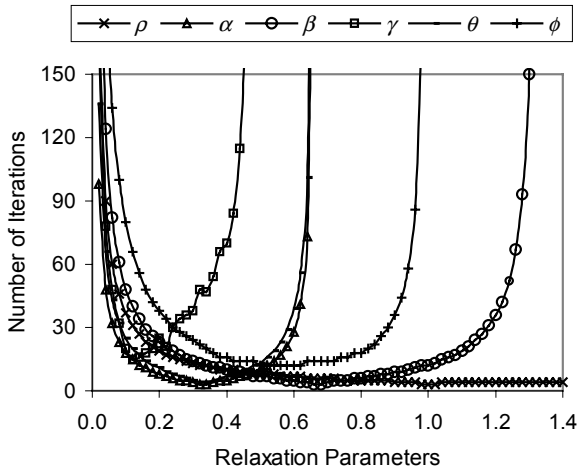


Figure 4: Applicable range and optimal values of the relaxation parameters ($a_B/a_F=1$ and $K_B/K_F=2$).

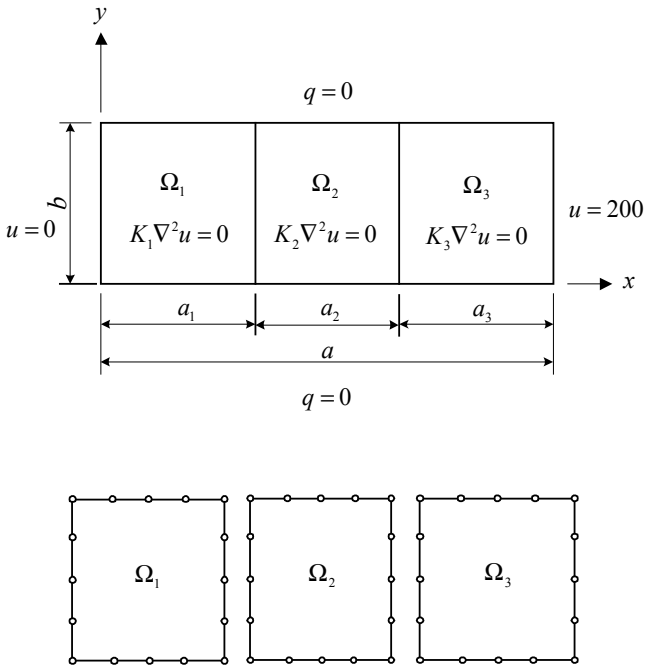
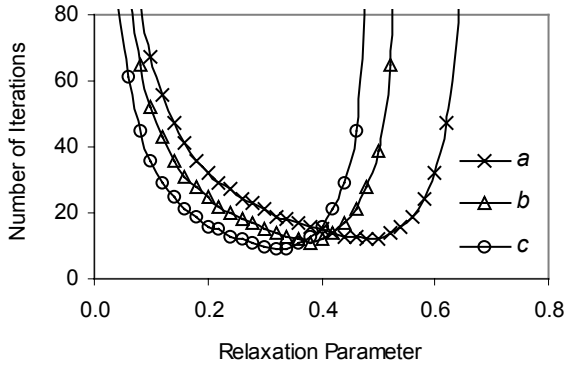


Figure 5: BEM-BEM coupling example, potential problem with the domain decomposed into three sub-domains.

As an BEM-BEM coupling example, let us reconsider the potential flow problem but with the domain decomposed into three sub-domains (Figure 5). The Neumann-Neumann interface relaxation algorithm is not suited for solving this problem. The problem was solved using the geometric contraction based BEM-BEM coupling algorithm. The results agree very well with the analytical solution. Figure 6 shows the applicable ranges and the optimal values of the relaxation parameter α for different combinations of K_1, K_2 and K_3. For example α should be within [0.02-0.54] to assure convergence and the optimal value is determined as 0.38, for $K_1=1.0$, $K_2=1.0$ and $K_3=2.0$.

Conclusions

In this paper, several new interface relaxation coupling algorithms are presented. The interface relaxation coupling algorithms make it easy to implement different software for different sub-domains. Further, they are easily extendable to solve nonlinear problems, which will exploit further possibilities of more complicated and large-scale problems. The wide class of the coupling algorithms, presented in this paper, seems to have the potential to work effectively. However there is still much to be learned about their behavior and about how to choose among them or to choose their parameters. These issues may be considered for future investigations.



a) K_1=1.0, K_2=1.0, K_3=1.0 b) K_1=1.0, K_2=1.0, K_3=2.0
c) K_1=2.0, K_2=1.0, K_3=2.0

Figure 6: Applicable ranges and the optimal values of the relaxation parameter α for different combinations of K_1, K_2 and K_3.

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References

1. Zienkiewicz, O. C., Kelly D. W. and Bettes P., "The Coupling of the Finite Element Method and Boundary Solution Procedures," *International Journal for Numerical Methods in Engineering*, Vol. 11, 1977, pp. 355-375.
2. State of the Art in BEM/FEM Coupling, *The International Journal of Boundary Element Methods Communications*, Vol. 4, No. 2, 1993, pp. 58-67.
3. State of the Art in BEM/FEM Coupling, *The International Journal of Boundary Element Methods Communications*, Vol. 4, No. 3, 1993, pp. 94-104.
4. Ganguly S, Layton J. B. and Balakrishma, C., "Symmetric coupling of multi-zone curved Galerkin boundary elements with finite elements in elasticity," *International Journal of Numerical Methods in Engineering*, Vol. 48, 2000, pp. 633-654.
5. Bialecki, R. A., Ostrowski Z., Kassab, A. J., Yin, Q. and Sciubba, E., "Coupling BEM, FEM and analytic solutions in steady-state potential problems," *Engineering Analysis with Boundary Elements*, Vol. 26, 2002, pp. 597-611.
6. Elleithy, W. M. and Al-Gahtani, H. J., "An Overlapping Domain Decomposition Approach for Coupling the Finite and Boundary Element Methods," *Engineering Analysis with Boundary Elements*, Vol. 24, No. 5, 2000, pp. 391-398.
7. Stein, E. and Kreienmeyer, M., "Coupling of BEM and FEM by a Multiplicative Schwarz Method and its Parallel Implementation," *Engineering Computations*, Vol. 15, No. 2, 1998, pp. 173-198.
8. Kamiya, N., Iwase, H. and Kita, E., "Parallel Computing for the Combination Method of BEM and FEM," *Engineering Analysis with Boundary Elements*, 18, 1996, pp. 221-229.
9. Lin, Chin-Ching, Lawton, E. C., Caliendo, J. A. and Anderson, L. R., "An Iterative Finite Element-Boundary Element Algorithm," *Computers & Structures*, Vol. 39, No. 5, 1996, pp. 899-909.
10. Feng, Y. T. and Owen, D. R. J., "Iterative Solution of Coupled FE/BE Discretization for Plate-Foundation Interaction Problems," *International Journal for Numerical Methods in Engineering*, Vol. 39, 1996, pp. 1889-1901.
11. Kamiya, N. Iwase, H. & Kita, E., "Parallel implementation of boundary element method with domain decomposition," *Engineering Analysis with Boundary Elements*, vol. 18, 1997, pp. 209-216.
12. Mu, M., "Solving composite problems with interface relaxation," *SIAM J. Sci. Comput.*, Vol. 20, No. 4, 1999, pp. 1394-1416.
13. Funaro, D., Quarteroni, A. and Zanolli, P., "An Interface Procedure with Interface Relaxation for Domain Decomposition Methods," *SIAM J. Numer. Anal.*, Volume 25, No. 6, 1988, pp. 1213-1236.
14. Lions, P. L., "On the Schwarz Alternating Method: A Variant for Nonoverlapping Subdomains," In Glowinski, R., Golub, G. H., Meurant, G. A. and Periaux, J., editors, *Domain Decomposition Methods for Partial Differential Equations*, SIAM, 1990, pp. 202-223.
15. Mu, M. and Rice, J. R., "Modeling with Collaborating PDE Solvers–Theory and Practice," *Computing Systems in Engineering*, Vol 6, 1995, pp. 87-95.
16. Rice, J. R., Tsompanopoulou P. and Vavalis, E. A., "Interface Relaxation Methods for Elliptic Differential Equations," *Applied Numerical Mathematics*, Vol. 32, 1999, pp. 219-245.
17. Elleithy, W. M. and Masataka Tanaka, "Interface Relaxation Algorithms for Coupling the FEM and BEM," *24th World Conference on the Boundary Element Method*, Sintra, Portugal, 2002, pp. 721-730.
18. El-Gebeily, M., Elleithy, W. M. and Al-Gahtani, H. J., "Convergence of the Domain Decomposition Finite Element-Boundary Element Coupling Methods," *Computer Methods in Applied Mechanics and Engineering*, Vol. 191, Issue 43, 2002, pp. 4851-4867.