Transactions of JASCOME Vol.5, No.2 (Dec. 2005) Paper No. 15-051209 **JASCOME**

Analysis of multi-crack problems in orthotropic elastic body based on fast multipole boundary element method

De Fa WANG[1)], Takuo FUKUI[2)], Takaaki KOBAYASHI[3)]

[1)] Graduate School of Engineering, University of Fukui (〒 910-8507 Bunkyo 3-9-1, Fukui City, E-mail: defawang@taku.anc-d.fukui-u.ac.jp)
[2)] Graduate School of Engineering, University of Fukui (〒 910-8507 Bunkyo 3-9-1, Fukui City, E-mail: tak@taku.anc-d.fukui-u.ac.jp)
[3)] Graduate School of Engineering, University of Fukui (〒 910-8507 Bunkyo 3-9-1, Fukui City, E-mail: yasshi@taku.anc-d.fukui-u.ac.jp)

This paper is concerned with the analysis of multi-crack problems in orthotropic elastic body by the Fast Multipole Boundary Element Method (FM-BEM). First, based on the complex potential functions, a hyper-singular boundary integral equation for crack problems is presented where a crack tip element is introduced to improve the accuracy of the crack solution. Then, fast multipole method is adopted to reduce the computational complexity of the boundary element method, for which the multipole moments of the influence functions of the crack-tip element are presented. The validation of the computing efficiencies of FM-BEM is carried out by numerical examples, which suggests that the size of the crack problem one can solve will increase dramatically with the use of FMM. Also, the crack displacement solutions are calculated successfully by FM-BEM for a large-scale example, which shows the applicability and the validity of this fast multipole boundary element method in solving large-scale crack problems for two-dimensional orthotropic materials.

Key Words : *Boundary element method, Fast multipole method, Orthotropic elasticity, Crack, Computational efficiency*

1. Introduction

It is well known that microcracks in brittle materials not only lead to macrocrack initiation and final failure, but also induce progressive damage. So, the study on the crack problems is of significant importance in engineering. For the fracture analysis, the boundary element method is now a well established numerical technique. The BEM is particularly efficient for crack problems due to its ability to model high stress gradients like those occurring near the crack tip, and due to simple remeshing procedures to study crack propagation. Despite the vast number of applications of the conventional BEM to the isotropic problems, few works with the FM-BEM are found in the analysis of the orthotropic crack problems and even fewer works for the general anisotropic problems by FM-BEM. For isotropic materials, Fukui[1)] has studied the two-dimensional, elastostatic problems with a large number of cracks based on the FM-BEM, from which the high efficiency of this method for large-scale crack problems has been proved.

However, the analysis on the multi-crack problems for orthotropic materials based on the FM-BEM, to the authors' knowledge, has not been considered. Therefore, the objective of this paper is to study the applications of FM-BEM to the large scale crack problems in orthotropic materials, from which the possibility to solve large-scale crack problems based on BEM is expected. Since we have presented a fast multipole boundary element method for two-dimensional orthotropic materials[2)], and numerical examples did show the high efficiency of this method in solving the large-scale problem for orthotropic materials, this work could be regarded as an extension of Fukui's work.[1)2)]

2. 2-D Orthotropic Elasticity

2.1 Boundary value problem

For two-dimensional orthotropic elastic materials, the strain-displacement relation, the equlibrium equation and the constitutive equation are given as:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1}$$

$$\sigma_{ij,j} + X_i = 0 \tag{2}$$

$$\sigma_{ij} = c_{ij}^{kl}\epsilon_{kl} \qquad \begin{pmatrix} c_{11}^{11} & c_{11}^{22} & 0 \\ c_{22}^{11} & c_{22}^{22} & 0 \\ 0 & 0 & 2c_{12}^{12} \end{pmatrix} \tag{3}$$

or

$$\epsilon_{ij} = s_{ij}^{kl}\sigma_{kl} \qquad \begin{pmatrix} s_{11}^{11} & s_{11}^{22} & 0 \\ s_{22}^{11} & s_{22}^{22} & 0 \\ 0 & 0 & 2s_{12}^{12} \end{pmatrix} \tag{4}$$

where ϵ_{ij}, u_i, σ_{ij} and X_i are strain, displacement, stress and body force, respectively. c_{ij}^{kl} and s_{ij}^{kl} are the elastic and compliance tensors, respectively.

Consider a boundary-value problem with multi-cracks where the domain and its boundary are expressed by B and $\partial B = \partial B_1 + \partial B_2$, respectively. $S_1, S_2, ..., S_M$ are the crack face boundaries. Then, based on the basic equations above, the crack problem can be described as

$$\begin{aligned} c_{ij}^{kl} u_{k,lj} + X_i = 0 & \quad (\text{in } B) \\ u_i = \hat{u}_i & \quad (\text{on } \partial B_1) \\ s_i = \sigma_{ij} n_j = \hat{s}_i & \quad (\text{on } \partial B_2) \\ s_i = \sigma_{ij} n_j^{\pm} = 0 & \quad (\text{on } S_1, S_2, ..., S_M) \end{aligned} \tag{5}$$

where n_i is the unit outward normal vector on ∂B. $n_i^{\pm}$ are the unit outward normal vectors on the both side surfaces of the crack, and $n_i = n_i^+ = -n_i^-$ is defined for the reference. s_i is the traction vector on the boundary. $\hat{u}_i$ and $\hat{s}_i$ are the given vectors on the boundary.

2.2 Solutions based on complex functions[3)]

Assume that the body forces are absent. Then, based on the Airy's stress function analysis, the governing equations for two-dimensional orthotropic elasticity can be derived as

$$S_{22}^{11} \frac{\partial^4 \psi}{\partial z^4} - 4 S_{12}^{11} \frac{\partial^4 \psi}{\partial z^3 \partial \bar{z}} + 2(S_{11}^{11} + 2S_{12}^{12}) \frac{\partial^4 \psi}{\partial z^2 \partial \bar{z}^2} - 4 S_{11}^{12} \frac{\partial^4 \psi}{\partial z \partial \bar{z}^3} + S_{11}^{22} \frac{\partial^4 \psi}{\partial \bar{z}^4} = 0 \tag{6}$$

where a bar denotes the complex conjugate. ψ is the stress function, and $z = x_1 + ix_2$. Parameters S_{ij}^{kl} are given as

$$\begin{aligned} S_{11}^{11} &= \frac{1}{4}\left(s_{11}^{11} + s_{22}^{22} + 4s_{12}^{12} - 2s_{11}^{22}\right) \\ S_{11}^{22} = S_{22}^{11} &= \frac{1}{4}\left(s_{11}^{11} + s_{22}^{22} - 4s_{12}^{12} - 2s_{11}^{22}\right) \\ S_{11}^{12} = S_{12}^{11} &= \frac{1}{4}\left(s_{11}^{11} - s_{22}^{22}\right) \\ S_{12}^{12} &= \frac{1}{4}\left(s_{11}^{11} + s_{22}^{22} + 2s_{11}^{22}\right) \end{aligned}$$

A solution of (6) of the form $\psi(z + \gamma\bar{z})$ exists provided the complex constant γ is a root of the characteristic equation of (6). Then, since ψ is a real function, the general solution of (6) can be expressed as

$$\psi(z) = \phi(z_1) + \overline{\phi}(\bar{z}_1) + \chi(z_2) + \overline{\chi}(\bar{z}_2) \tag{7}$$

where $z_\alpha = z + \gamma_\alpha \bar{z}$ $(\alpha = 1, 2)$, while ϕ and χ are two analytical functions. For orthotropic bodies, γ_1 and γ_2 are either complex conjugates or real.

If we make α_λ^2 be the two roots of the following equation

$$s_{22}^{22} \alpha^4 - 2(s_{22}^{11} + 2s_{12}^{12})\alpha^2 + s_{11}^{11} = 0 \tag{8}$$

the characteristic roots γ_1 and γ_2 can be obtained by

$$\gamma_\lambda = \frac{\alpha_\lambda - 1}{\alpha_\lambda + 1}, \quad \alpha_\lambda = \frac{1 + \gamma_\lambda}{1 - \gamma_\lambda} \quad (\lambda = 1, 2) \tag{9}$$

Then, based on (7), the displacement and the stress fields in orthotropic materials can be derived as follows

$$D = u_1 + iu_2 = \delta_1 \phi'(z_1) + \rho_1 \overline{\phi}'(\bar{z}_1) + \delta_2 \chi'(z_2) + \rho_2 \overline{\chi}'(\bar{z}_2) \tag{10}$$

$$\Phi = \sigma_{11} - \sigma_{22} + 2i\sigma_{12} = -4\gamma_1^2 \phi''(z_1) - 4\overline{\phi}''(\bar{z}_1) - 4\gamma_2^2 \chi''(z_2) - 4\overline{\chi}''(\bar{z}_2) \tag{11}$$

$$\Theta = \sigma_{11} + \sigma_{22} = 4\gamma_1 \phi''(z_1) + 4\overline{\gamma}_1 \overline{\phi}''(\bar{z}_1) + 4\gamma_2 \chi''(z_2) + 4\overline{\gamma}_2 \overline{\chi}''(\bar{z}_2) \tag{12}$$

where δ_α and ρ_α are parameters associated with the orthotropic material constants which are given as

$$\begin{aligned} \delta_1 &= (1 + \gamma_1)\beta_2 - (1 - \gamma_1)\beta_1 \\ \delta_2 &= (1 + \gamma_2)\beta_1 - (1 - \gamma_2)\beta_2 \\ \overline{\rho_1} &= (1 + \gamma_1)\beta_2 + (1 - \gamma_1)\beta_1 \\ \overline{\rho_2} &= (1 + \gamma_2)\beta_1 + (1 - \gamma_2)\beta_2 \end{aligned} \tag{13}$$

where $\beta_\lambda = s_{22}^{11} - s_{22}^{22}\alpha_\lambda^2$ $(\lambda = 1, 2)$, which is valid only for orthotropic bodies.

3. Boundary Element Method

3.1 Boundary integral equations

From Somigliana's identity, the displacement solutions of the crack problem (5) can be expressed as

$$C_{ij}(\boldsymbol{x}) u_j(\boldsymbol{x}) = \mathring{u}_i(\boldsymbol{x}) + \int_{\partial B} G_{ij}(\boldsymbol{x}, \boldsymbol{y}) s_j(\boldsymbol{y})\, dS_y - \int_{\partial B} S_{ij}(\boldsymbol{x}, \boldsymbol{y}) u_j(\boldsymbol{y})\, dS_y - \sum_K \int_{S_K} S_{ij}(\boldsymbol{x}, \boldsymbol{y}) [u_j](\boldsymbol{y})\, dS_y \tag{14}$$

where C_{ij} is a parameter depending on the location of point $\boldsymbol{x}$. If the boundary is smooth at the point $\boldsymbol{x}$, C_{ij} is determined as follows: $C_{ij} = \delta_{ij}$ when $\boldsymbol{x} \in B$; $C_{ij} = 0$ when $\boldsymbol{x} \notin B + \partial B$; When $\boldsymbol{x} \in \partial B$, $C_{ij} = \delta_{ij}/2$. $\mathring{u}_i$ is the term due to the body force. $[u_i] = u_i^+ - u_i^-$ is the crack opening displacement. The kernels G_{ij} and S_{ij} are the fundamental solution and the associated fundamental solution, respectively, which can be expressed in terms of complex functions[2)].

From (14), we can get the expression for the traction vectors on the crack surface as

$$0 = n_j \sigma_{ji}^0 + T_{ij}^n \mathring{u}_j(\boldsymbol{x}) + \int_{\partial B} \tilde{S}_{ij}(\boldsymbol{x}, \boldsymbol{y}) s_j(\boldsymbol{y})\, dS_y - \int_{\partial B} U_{ij}(\boldsymbol{x}, \boldsymbol{y}) u_j(\boldsymbol{y})\, dS_y - \sum_K \mathrm{Pf} \int_{S_K} U_{ij}(\boldsymbol{x}, \boldsymbol{y}) [u_j](\boldsymbol{y})\, dS_y \tag{15}$$

where σ_{ji}^0 is the initial stress. $\tilde{S}_{ij}(\boldsymbol{x}, \boldsymbol{y})$ is the adjoint of $S_{ij}(\boldsymbol{x}, \boldsymbol{y})$. The kernels $\tilde{S}_{ij}(\boldsymbol{x}, \boldsymbol{y})$ and $U_{ij}(\boldsymbol{x}, \boldsymbol{y})$ can be derived by

$$\tilde{S}_{ij}(\boldsymbol{x}, \boldsymbol{y}) = T_{ik}^x G_{kj}(\boldsymbol{x}, \boldsymbol{y}), \quad U_{ij}(\boldsymbol{x}, \boldsymbol{y}) = T_{ik}^x S_{kj}(\boldsymbol{x}, \boldsymbol{y}) \tag{16}$$

where T_{ij}^x is the operator to define the traction vectors on the boundary point $\boldsymbol{x}$ through the displacement vectors. The kernel $U_{ij}(\boldsymbol{x}, \boldsymbol{y})$ is hyper-singular when $\boldsymbol{x} = \boldsymbol{y}$, so the integral that includes this kernel is divergent and must be evaluated as the finite part that is denoted by the symbol ⌈Pf⌋.

In this paper, only the infinite elastic field is considered, and the body force components are assumed to be absent. Therefore, only the crack face must be considered. Then, Eq.(15) becomes

$$0 = n_j\sigma^0_{ji} - \sum_{K=1}^{M} \mathrm{Pf}\int_{S_K} U_{ij}(\boldsymbol{x},\boldsymbol{y})[u_j](\boldsymbol{y})\,dS_y \tag{17}$$

3.2 Discretization of boundary integral equations

As in Fukui's work [2)], Eq.(17) can be discretized into the following system:

$$0 = s_i^{0I} - \sum_{K=1}^{M}\sum_{I}^{N_K} D^I_{ij}(\boldsymbol{x})[u_j]^I \tag{18}$$

where $s_i^{0I} = n_j\sigma^0_{ji}(\boldsymbol{x}_I)$, and

$$D^I_{ij}(\boldsymbol{x}) = \int_{E_I} U_{ij}(\boldsymbol{x},\boldsymbol{y})f_I(\boldsymbol{y})\,ds_y \tag{19}$$

where $f_I(\boldsymbol{y})$ is the basis function. For the constant density element, $f_I(\boldsymbol{y}) = 1$. The influence functions of the constant density element have been discussed by Fukui[2)]. Because the displacement field near the crack tip is in proportion to $\sqrt{s}$ where s is the distance from the crack tip [4)], the constant density element, as we can see, is not suitable for the crack tip area. Therefore, in order to improve the calculating accuracy, a crack tip element is introduced in this paper and will be discussed in the following section.

3.3 Crack tip element

As shown in **Fig.1**, the displacement components near the crack tip are in proportion to $\sqrt{s}$ [4)]. Therefore, the basis function $f_I(\boldsymbol{y})$ should be also in proportion to $\sqrt{s}$. Also, to be consistent with the constant density element, the value of function $f_I(\boldsymbol{y})$ on the middle point of the crack tip element should be 1, so we define the basis function as

$$f_I(\boldsymbol{x}) = \sqrt{2s/a} \tag{20}$$

where a is the length of the crack tip element. The influence function (19) becomes

$$D^I_{ij}(\boldsymbol{x}) = \int_{E_I} U_{ij}(\boldsymbol{x},\boldsymbol{y})\sqrt{2s/a}\,ds_y \tag{21}$$

Now, we will discuss the derivation of the influence function $D^I_{ij}(\boldsymbol{x})$.

3.4 Stress fields due to crack-tip element

For the elasic orthotropic materials, the complex functions due to the associated fundamental solution, i.e. ϕ^S and χ^S, are given as[2)]

$$\phi^S(z_1) = \frac{V}{2\pi}\log z_1, \quad \chi^S(z_2) = \frac{W}{2\pi}\log z_2 \tag{22}$$

where V and W are given as

$$V = \frac{(1+\alpha_1)}{4(1+\gamma_1)(\alpha_1^2-\alpha_2^2)s_{22}^{22}}\,\hat{\nu}_1\hat{U}_1 \tag{23}$$

$$W = \frac{-(1+\alpha_2)}{4(1+\gamma_2)(\alpha_1^2-\alpha_2^2)s_{22}^{22}}\,\hat{\nu}_2\hat{U}_2 \tag{24}$$

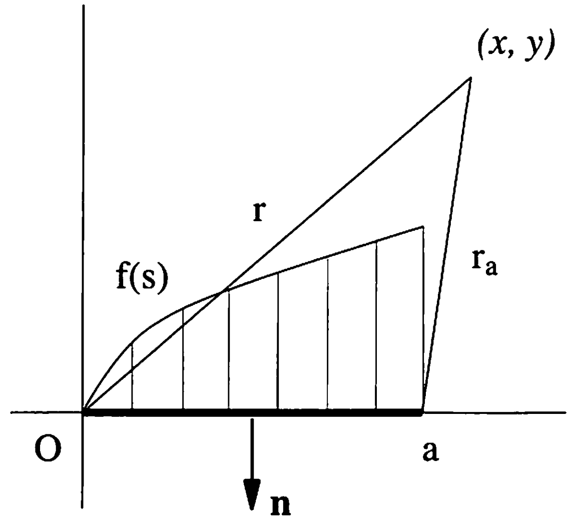

Fig.1 Crack-tip element

where $U = U_1 + iU_2$ is the dislocation, $\nu = n_1 + in_2$, $\hat{\nu}_\lambda = \nu - \gamma_\lambda\overline{\nu}$ and $\hat{U}_\lambda = U - \gamma_\lambda\overline{U}$.

Define a crack-tip element in the complex plane, as shown in **Fig.2**, in which the crack tip is at the origin O. Since the length s is given by

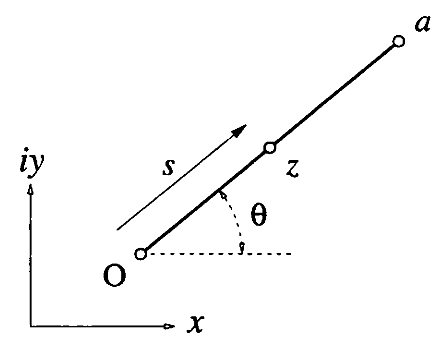

Fig.2 Crack-tip element in complex plane

$$s = \frac{z_\alpha}{e^{i\theta}+\gamma_\alpha e^{-i\theta}} = \frac{z_\alpha}{\tau_\alpha}, \tag{25}$$

we have

$$ds = \frac{dz_\alpha}{e^{i\theta}+\gamma_\alpha e^{-i\theta}} = \frac{dz_\alpha}{\tau_\alpha} \tag{26}$$

Thus (20) becomes

$$f_I(\boldsymbol{x}) = \sqrt{2s/a} = \sqrt{\frac{2z_\alpha}{a\tau_\alpha}} \tag{27}$$

where $\sqrt{z}$ implies $z^{1/2}$.

Assume that $z \doteq \boldsymbol{x}$、$\zeta \doteq \boldsymbol{y}$. For the stress fields due to the crack-tip element, the following integral included in the integral with the hyper-singular kernel U_{ij} must be evaluated:

$$\int_E \phi^{S''}(z_1-\zeta_1)f(s)\,ds_y = \sqrt{\frac{2}{a\tau_1^3}}\int_0^{a_1}\phi^{S''}(z_1-\zeta_1)\sqrt{\zeta_1}\,d\zeta_1 \tag{28}$$

Since $\phi^{S'}(z-\zeta) = [1/(z-\zeta)](V/2\pi)$, we have

$$\phi^{S''}(z-\zeta) = [-1/(z-\zeta)^2](V/2\pi) \tag{29}$$

Then, we can get the integration of functions $\phi^{S''}$ and $f(s)$ as

$$\int_E \phi^{S''}(z_1-\zeta_1)f(s)\,ds_y = \frac{V}{2\pi}\sqrt{\frac{2}{a\tau_1^3}}\left(\frac{1}{2\sqrt{z_1}}\log\frac{\sqrt{z_1}+\sqrt{a_1}}{\sqrt{z_1}-\sqrt{a_1}}-\frac{\sqrt{a_1}}{z_1-a_1}\right) \tag{30}$$

Similarly, we can get the integration of $\chi^{S''}$ with $f(s)$.

Then, the complex stress fields due to a crack-tip element become

$$\begin{aligned}\int_E \Phi^S(\boldsymbol{x},\boldsymbol{y})\,ds_y &= U_j\int_E(\sigma_{11}^j-\sigma_{22}^j+2i\sigma_{12}^j)\,ds_y\\ &= -4[\gamma_1^2Vk_1(z_1)+\overline{Vk}_1(\bar{z}_1)\\ &\quad+\gamma_2^2Wk_2(z_2)+\overline{Wk}_2(\bar{z}_2)]\end{aligned} \tag{31}$$

$$\begin{aligned}\int_E \Theta^S(\boldsymbol{x},\boldsymbol{y})\,ds_y &= U_j\int_E(\sigma_{11}^j+\sigma_{22}^j)\,ds_y\\ &= 4[\gamma_1Vk_1(z_1)+\bar{\gamma}_1\overline{Vk}_1(\bar{z}_1)\\ &\quad+\gamma_2Wk_2(z_2)+\bar{\gamma}_2\overline{Wk}_2(\bar{z}_2)]\end{aligned} \tag{32}$$

where Φ^S and Θ^S are the complex stress components due to the associated fundamental solution, respectively. $k_\alpha(z)$ is defined by

$$k_\alpha(z)=\frac{1}{2\pi}\sqrt{\frac{2}{a\tau_\alpha^3}}\left(\frac{1}{2\sqrt{z}}\log\frac{\sqrt{z}+\sqrt{a_\alpha}}{\sqrt{z}-\sqrt{a_\alpha}}-\frac{\sqrt{a_\alpha}}{z-a_\alpha}\right) \tag{33}$$

3.5 Kernel U_{ij} and influence function due to crack-tip element

As discussed above, the hyper-singular kernel $U_{ij}(\boldsymbol{x},\boldsymbol{y})$ is the traction field at $\boldsymbol{x}$ due to the double layer kernel, which can be expressed by

$$\begin{aligned}T(\boldsymbol{x},\boldsymbol{y}) &= T_1+iT_2 = U_{1j}(\boldsymbol{x},\boldsymbol{y})U_j+iU_{2j}(\boldsymbol{x},\boldsymbol{y})U_j\\ &= \frac{1}{2}\left(\Theta^S\nu^x+\Phi^S\bar{\nu}^x\right)\end{aligned} \tag{34}$$

where ν^x is defined by $\nu^x = n_1^x+in_2^x$ at $\boldsymbol{x}$. Then, without in details, the influence function of $D_{ij}(\boldsymbol{x},\boldsymbol{y})$ due to a crack-tip element can be expressed as

$$\begin{aligned}&\int_E T(\boldsymbol{x},\boldsymbol{y})f(s)\,ds_y\\ &=U_j\left[\int_E U_{1j}(\boldsymbol{x},\boldsymbol{y})f(s)\,ds_y+i\int_E U_{2j}(\boldsymbol{x},\boldsymbol{y})f(s)\,ds_y\right]\\ &= 2\,[\nu_1^x\gamma_1Vk_1(z_1)-\bar{\nu}_1^x\overline{Vk}_1(\bar{z}_1)\\ &\quad+\nu_2^x\gamma_2Wk_2(z_2)-\bar{\nu}_2^x\overline{Wk}_2(\bar{z}_2)]\end{aligned} \tag{35}$$

where ν_α^x is defined by $\nu_\alpha^x = \nu^x-\gamma_\alpha\bar{\nu}^x$. From (35), the influence functions due to the crack tip element can be obtained, from which the multi-crcak boundary value probelms can be solved numerically.

4. Fast Multipole Method

In this paper, the Fast Multipole Method (FMM) is adopted to reduce the computational complexity of the multi-crack problems. To implement this fast method, the multipole expansion, the local expansion and the translation formulas are necessary, which were discussed in detail in Fukui's work[2)] and omitted here. In this section only the multipole moment due to the crack-tip element will be discussed.

For the elasic orthotropic materials, the general form of the multipole expansion of functions ϕ^S and ϕ^S can be given as[2)]

$$\phi^S(z_1-\zeta_1) = \frac{1}{2\pi}\left[-M_0\log z_1+\sum_{n=1}^{\infty}\frac{M_n}{z_1^n}\right] \tag{36}$$

$$\chi^S(z_2-\zeta_2) = \frac{1}{2\pi}\left[-N_0\log z_2+\sum_{n=1}^{\infty}\frac{N_n}{z_2^n}\right] \tag{37}$$

Based on (36) and (37), we can get the multipole moments of the associated fundamental solution as

$$M_0^S = -V, \qquad M_n^S = -\frac{V\zeta_1^n}{n} \tag{38}$$

$$N_0^S = -W, \qquad N_n^S = -\frac{W\zeta_2^n}{n} \tag{39}$$

Consider a crack tip element $[0,b]$, as shown in **Fig.3**.

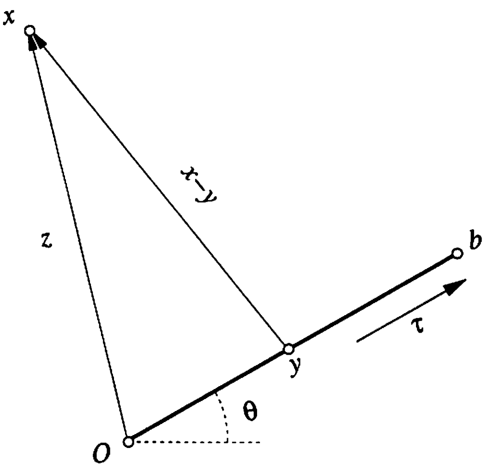

Fig.3 Crack tip element and multipole point

Origin O is at the crack tip. Then we have

$$\sqrt{s/a} = \sqrt{\zeta_\alpha/b_\alpha} \tag{40}$$

By integrating the multipole coefficients M_n^S and N_n^S through the crack-tip element, e.g.,

$$\begin{aligned}\tilde{M}_n^S &= \int_E M_n^S(\zeta_1)\sqrt{2s/a}\,ds_y\\ &= \frac{\sqrt{2}}{\tau_1\sqrt{b_1}}\int_0^{b_1}M_n^S(\zeta_1)\sqrt{\zeta_1}\,d\zeta_1\end{aligned} \tag{41}$$

we can get the multipole moments $\tilde{M}_n^S$ and $\tilde{N}_n^S$ of the crack tip element due to the double layer kernel as

$$\tilde{M}_0^S = \frac{-2\sqrt{2}V}{3\tau_1}b_1, \quad \tilde{M}_n^S = \frac{-\sqrt{2}V}{n(n+\frac{3}{2})\tau_1}b_1^{n+1} \tag{42}$$

$$\tilde{N}_0^S = \frac{-2\sqrt{2}W}{3\tau_2}b_2, \quad \tilde{N}_n^S = \frac{-\sqrt{2}W}{n(n+\frac{3}{2})\tau_2}b_2^{n+1} \tag{43}$$

where $\tau_\alpha = e^{i\theta} + \gamma_\alpha e^{-i\theta}$. After the multipole moments of the crack tip element has been derived, the fast multipole method can be implemented in BEM for orthotropic large-scale crack problems.

5. Validation of Computing Efficiencies

In this section, we will give two numerical examples to examine the computing efficiencies of the fast multipole boundary element method for the orthotropic crack problems. Here, we consider relatively simple problems using, possibly, more elements than what would be necessary for engineering purpose, since the objective of this study is to test the performance of the proposed method rather than to solve the practical problems. For example, the maximum number of the boundary elements in the examples is chosen to be about 126,000.

In the following examples, the number of terms in the multipole expansions is truncated at $n = 25$. Also, the maximum number of boundary elements in a leaf is chosen to be 8. For these two examples, the material constants are adopted as follows [2)]

$$E_1 = E, \quad E_2/E_1 = 2, \quad \nu_{12} = 0.25, \quad G_{12}/E_1 = 0.4$$

The first example is, as shown in **Fig.4**, about a finite crack considered in an infinite orthotropic elastic body. Its orientation angle is 45° with respect to x_1 axis, and its half length is $a = 1$.

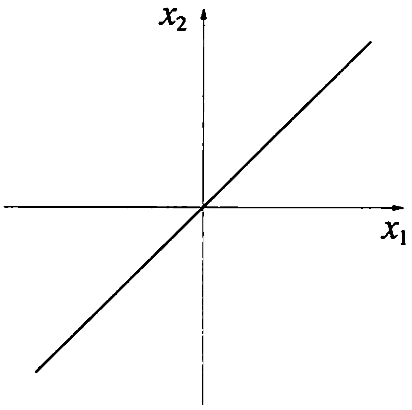

Fig.4 An inclined finite crack in an infinite orthotropic body

The second example is about 3 × 3 finite cracks considered in an infinite orthotropic elastic body, as shown in **Fig.5**. Every crack has the same orientation angle, i.e. 45° with respect to x_1 axis, and has the same half length $a = 1$. The distance between the centres of two neighbouring cracks is assumed to be 4.

The Jacobi method was adopted as the iteration solver in this paper. The numeber of iterations for these two examples is 517 and 981, respectively. For these two examples, the numerical results for the computing time and the used memory are shown in **Fig.6** and **Fig.7**, respectively.

Fig.6 plots the computing time per iteration versus the number of elements for these two examples, from which we can see that, the computing time is in pro-

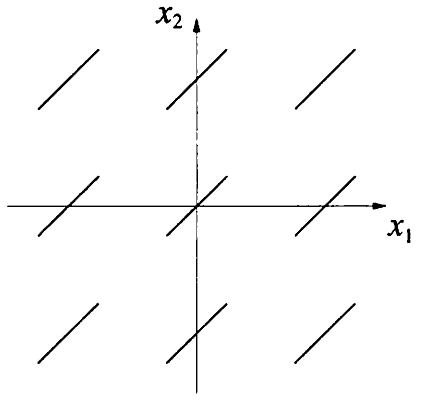

Fig.5 3 × 3 inclined finite cracks in an infinite orthotropic body

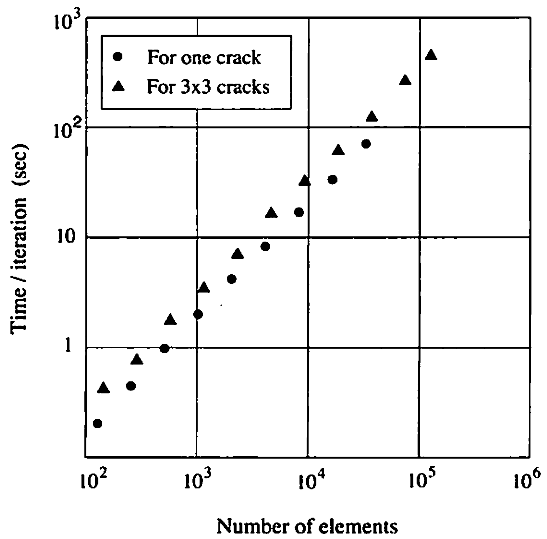

Fig.6 Computing time per iteration versus the number of elements

portion to the number of the elements when the fast multipole method is adopted.

Fig.7 plots the used memory versus the number of elements for these two examples, from which we can see that, the used memory is also in proportion to the number of the elements as expected when the number of elements is larger than about 1000. From the results of these two examples, we can see that, both the computing time and the used memory are in proportion to the number of the elements as expected, from which the validation of the computing efficiencies of the presented FMBEM for orthotropic large-scale crack problems can be proved.

6. Computation of Crack Solutions

After the high efficiency of the fast multipole boundary element method for orthotropic crack problems has been proved, we will give a large-scale crack example to compute the crack solutions in this section.

The example is about 10 × 10 cracks in an infinite elastic orthotropic body, as shown in **Fig.8**. Assume that all the cracks have the same half-length, i.e., $a = 1$, and have the same inclined angle, i.e., 60° with respect

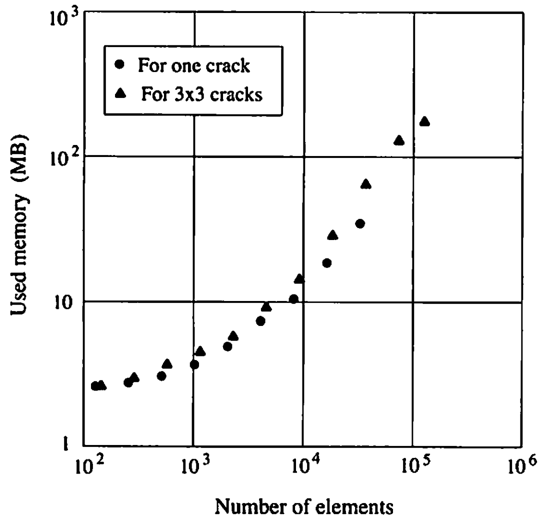

Fig.7 Used memory versus the number of boundary elements

to the x_1 axis. The distance between the centers of two neighbouring cracks is 3. The unidirectional tensile load in the x_2 direction is considered, and the same material constants as those in the last section are adopted.The number of boundry elements on each crack is 100, i.e., the total number of the elements in this example is 10,000. For this example, the displacement solutions on all the crack faces have been calculated successfully and are shown in **Fig.9**.

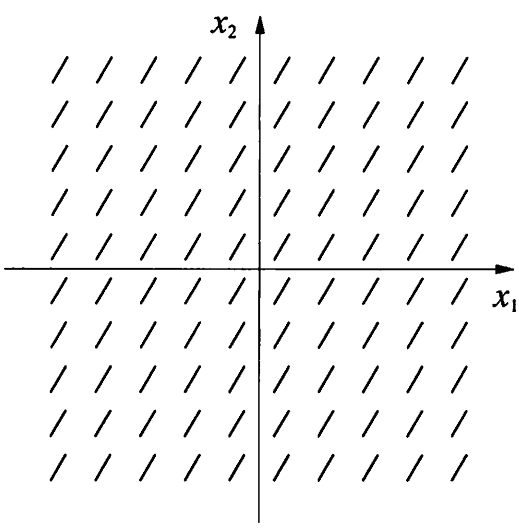

Fig.8 10 × 10 **finite cracks in an infinite orthotropic body**

7. Conclusions

Based on the analysis above, the work in this paper can be summarized as follows:

1. For two-dimensional orthotropic materials, a hyper-singular boundary integral equation for crack problems has been presented based on the complex potential functions, in which a crack tip element is specially introduced to improve the accuracy of the crack solution.

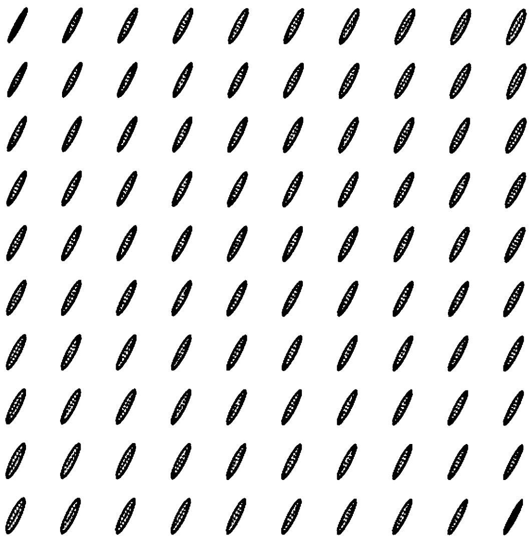

Fig.9 Displacement solutions on the 10 × 10 **finite crack faces**

2. FMM (fast multipole method) is adopted to reduce the computational complexity of BEM. The multipole moments of the influence functions of the crack-tip element have been presented. After discretizing the hyper-singular boundary integral equation by the collocation method, it can be solved numerically in connection with FMM.
3. Numerical results show that both the computational time and the used memory are in proportion to the number of the boundary elements, from which the validation of the computing efficiencies of FM-BEM has been proved. It can be concluded that the FM-BEM can decrease the computational complexities efficiently as expected, which suggests that the size of the crack problem one can solve will increase dramatically with the use of FMM.
4. The crack solutions have been calculated successfully by FM-BEM for a large-scale crack example, which shows the applicability and the validity of this fast multipole boundary element method for large-scale crack problems in orthotropic materials.

REFERENCES

1) Fukui, T., Mochida, T. and Inoue, K.: On crack extension analysis by boundary element method, *Journal of Boundary Element Method*, Vol.14, pp.47–52, 1997.
2) Fukui, T., Wang D.F. and Kobayashi, T.: Analysis of two-dimensional orthotropic elastostatic problem by fast multipole boundary element method, *Journal of Boundary Element Method*, Vol.21, pp.85–90, 2004.
3) Green, A.E. and Zerna, W.: *Theoretical Elasticity*, 2nd ed. Oxford, 1968.
4) Liebowitz, H.: Fracture, *Vol.II*, Academic Press, London, pp.123-126, 1968.