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A mode reduction method used for calculating the frequency response and topological derivatives

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For the calculation of elastic structure frequency response, a commonly used method is the full mode method (FM). However, for a frequency range, this approach tends to be computationally expensive, especially in the process of topology optimization. Therefore, this study proposes to use the mode displacement method (MDM), which is one of the mode reduction methods. Similarly, for topology optimization in the frequency range, using this method to calculate the adjoint operator can greatly reduce the calculation cost of topological derivatives. The results show that in frequency range, using MDM can significantly improve the efficiency of calculating frequency response and topological derivatives while ensuring a certain accuracy.

Key Words: Mode reduction, Mode displacement method, Frequency response, Topological derivative

1. Introduction

In numerical calculation of the response of a structure to a single frequency, a commonly used method is called the full mode method (FM), which provides quite accurate results. However, when calculating the response in a frequency interval, the calculation cost is very expensive because the response of different frequencies needs to be repeatedly calculated, and all modes of the discrete structure will be used each time. In practical situations, we are often interested in the low-frequency response. Therefore, mode reduction methods are becoming increasingly popular. These methods utilize only a few modes of the discrete structure, thereby reducing computational cost while maintaining accuracy. The mode displacement method (MDM) is one of the mode reduction methods used in this paper. It has a significant advantage when applied to the calculation of frequency band responses. When calculating multiple frequency responses with MDM, the eigenfrequencies and eigenvectors only need to be calculated once. Each subsequent frequency responses calculation involves only simple superposition, which significantly reduces the computational cost compared to FM.

In the context of frequency domain topology optimization, mode reduction methods are often applied. The first type is based on modal superposition，Ma[(1)] used MDM combined with the density-based method to perform optimization of the elastic structure. Liu[(2)] compared the optimization of MDM and mode acceleration method (MAM)[(3)]. Among them, MAM has more calculation cost than MDM, but its accuracy is slightly higher. The second type is based on series expansion, such as Jensen[(4)] used Padé expansion and density-based method. Yoon[(5)] used the Quasi-Static Ritz Vector (QSRV) to make the optimization.

The topological derivative is a new sensitivity analysis method, which is different from the previous gradient-based sensitivity analysis. Otomori[(6)], Filho[(7)] used topological derivatives to optimize static problems. Lopes[(8)] used a method based on topological derivatives to perform multi-load topology optimization of static problems. Giusti[(9)] used the topological derivative for topology optimization of anisotropic materials. Isakari[(10)] derived the topological derivative of the eigenvalue objective function based on the Helmholtz equation structure. Yamada[(11)] used this approach to optimize the response at specific frequency. When performing sensitivity analysis for non-self-adjoint situations, it is also necessary to calculate the

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adjoint operator of the topological derivative. The adjoint operator is usually obtained by solving the adjoint problem, which is also a computationally intensive process. In previous research, Hoshuku(12) used Padé approximation to approximate frequency response and topological derivative. In this study, MDM is also proposed for solving the adjoint problem of frequency band responses.

Overall, this paper aims to explore efficient methods for calculating the frequency response and topological derivatives of elastic structures. Using MDM for frequency response calculation and combining it with topological derivative to achieve efficient frequency band optimization. Numerical experiments will be performed to verify the effectiveness and efficiency of the proposed method, and comparisons with the FM method will be presented. The results of this study contribute to the development of improved optimization techniques for elastic structures under harmonic excitation.

2. Methods for Frequency Response Calculation

The forward problem calculation in this paper, which involves the frequency response calculation of the structure, will be based on the finite element method (FEM).

2.1. Full Mode Method (FM)

First, let's introduce the full mode method. For a linear elastic structure under harmonic excitation, as shown in Figure 1, it satisfies the following governing equations:

$$C_{ijkl}u_{k,lj} + \rho\omega^2 u_i = 0 \qquad \text{in } \Omega \tag{1}$$

$$t_i = \bar{t}_i \qquad \text{on } \Gamma_\mathrm{t} \tag{2}$$

$$u_i = \bar{u}_i \qquad \text{on } \Gamma_\mathrm{u}, \tag{3}$$

where C_{ijkl} is elasticity tensor, u_i is frequency response, ρ is material density, t_i is traction, $\bar{t}_i$ is the amplitude of the external force on boundary Γ_t, $\bar{u}_i$ is the specified displacement value on boundary Γ_u.

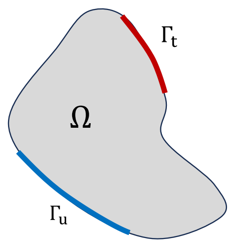


Fig. 1 Elastic body problem

The weak forms of (1) to (3) can be expressed as:

$$\int_\Omega \tilde{u}_{i,j} C_{ijkl} u_{k,l}\, \mathrm{d}\Omega - \omega^2 \int_\Omega \rho \tilde{u}_i u_i\, \mathrm{d}\Omega = \int_{\Gamma_t} \bar{t}_i \tilde{u}_i \mathrm{d}\Gamma, \tag{4}$$

here $\tilde{u}_i$ is the test function. The matrix form of (4) is:

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{u} = \mathbf{F}, \tag{5}$$

where $\mathbf{K}$ is stiffness matrix, $\mathbf{M}$ is mass matrix, $\mathbf{u}$ is the vector of frequency response, $\mathbf{F}$ is load vector. Through (5), the frequency response $\mathbf{u}$ can be obtained:

$$\mathbf{u} = (\mathbf{K} - \omega^2\mathbf{M})^{-1}\mathbf{F}. \tag{6}$$

It can be seen from (6) that for a system with n degrees of freedom, obtaining its response requires the inversion operation of an $n \times n$ matrix. This process requires a lot of calculations, and computational cost increases cubically as the degree of freedom of the system increases.

2.2. Mode Displacement Method (MDM)

The Mode Displacement Method is one of the mode reduction methods, which can efficiently calculate the displacement response of a structure with reduced computational effort. It is particularly useful for problems involving multiple frequency responses, significantly reducing computation time. For a linear elastic structure subjected to harmonic loads, it satisfies the following equation:

$$C_{ijkl}u_{k,lj} - \rho\ddot{u}_i = 0 \qquad \text{in } \Omega \tag{7}$$

$$t_i = \bar{t}_i e^{j\omega t} \qquad \text{on } \Gamma_\mathrm{t} \tag{8}$$

$$u_i = \bar{u}_i \qquad \text{on } \Gamma_\mathrm{u}, \tag{9}$$

where $\ddot{u}_i$ is the second derivative of displacement to time. The weak form of (7) to (9) is:

$$\int_\Omega \tilde{u}_{i,j} C_{ijkl} u_{k,l}\, \mathrm{d}\Omega + \int_\Omega \rho \tilde{u}_i \ddot{u}_i\, \mathrm{d}\Omega = \int_{\Gamma_t} \bar{t}_i e^{j\omega t} \tilde{u}_i \mathrm{d}\Gamma. \tag{10}$$

The matrix form of (10) is:

$$\mathbf{Ku} + \mathbf{M\ddot{u}} = \bar{\mathbf{T}} e^{j\omega t}. \tag{11}$$

For a discrete system with n degree of freedoms, its normalized i-th order eigenvector $\boldsymbol{\varphi}_i$ satisfies:

$$\boldsymbol{\varphi}_i^\mathrm{T}\mathbf{M}\boldsymbol{\varphi}_i = 1 \tag{12}$$

$$\boldsymbol{\varphi}_i^\mathrm{T}\mathbf{K}\boldsymbol{\varphi}_i = \omega_i^2, \tag{13}$$

where ω_i represents the i-th eigenvalue of the system. For the displacement vector, it can be expressed using the eigenvector matrix $\boldsymbol{\Phi}$ as the base conversion matrix, represented by generalized coordinates y_i:

$$\mathbf{u} = \boldsymbol{\Phi}\mathbf{y} = \sum_{i=1}^{n} \boldsymbol{\varphi}_i y_i, \tag{14}$$

where matrix $\boldsymbol{\Phi} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_\mathrm{n}]$. By substituting (12), (13) and (14) into the matrix form of the governing equation (11), we obtain:

$$\boldsymbol{\Phi}^\mathrm{T}\mathbf{M}\boldsymbol{\Phi}\ddot{\mathbf{y}} + \boldsymbol{\Phi}^\mathrm{T}\mathbf{K}\boldsymbol{\Phi}\mathbf{y} = \ddot{\mathbf{y}} + \mathrm{diag}(\omega_i^2)\mathbf{y} = \boldsymbol{\Phi}^\mathrm{T}\bar{\mathbf{T}} e^{j\omega t}. \tag{15}$$

Since the load $\bar{\mathbf{T}} e^{j\omega t}$ is harmonic, the response in generalized coordinates $\ddot{y}_i$ can also be obtained:

$$\ddot{y}_i = -\omega^2 y_i. \tag{16}$$

Substituting (16) into (15), we can obtain:

$$y_i = (\omega_i^2 - \omega^2)^{-1} \boldsymbol{\varphi}_i^\mathrm{T}\bar{\mathbf{T}} e^{j\omega t}. \tag{17}$$

Substituting (17) back into (14), obtained:

$$\mathbf{u} = \sum_{i=1}^{n} \boldsymbol{\varphi}_i y_i = \sum_{i=1}^{n} \frac{\boldsymbol{\varphi}_i^\mathrm{T}\bar{\mathbf{T}}\boldsymbol{\varphi}_i}{\omega_i^2 - \omega^2} e^{j\omega t}. \tag{18}$$

Therefore, the response amplitude should be:

$$\mathbf{u}' = \sum_{i=1}^{n} \frac{\boldsymbol{\varphi}_i^{\mathrm{T}} \bar{\mathbf{T}} \boldsymbol{\varphi}_i}{\omega_i^2 - \omega^2}. \tag{19}$$

By reducing the modes and considering only the first l-th eigenvalues and eigenvectors, we can get:

$$\mathbf{u}' = \sum_{i=1}^{l} \frac{\boldsymbol{\varphi}_i^{\mathrm{T}} \bar{\mathbf{T}} \boldsymbol{\varphi}_i}{\omega_i^2 - \omega^2}, \tag{20}$$

where $l \ll n$. The above formula represents the frequency response calculation using the MDM.

2.3. Comparison between two methods

Figure 2 and Figure 3 are the processes of the two frequency response calculation methods respectively.

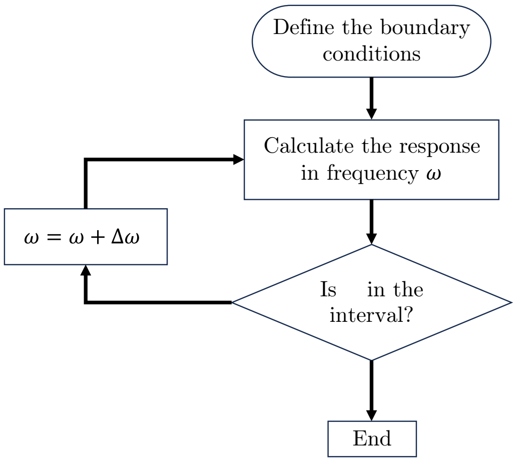


Fig. 2 Process of FM

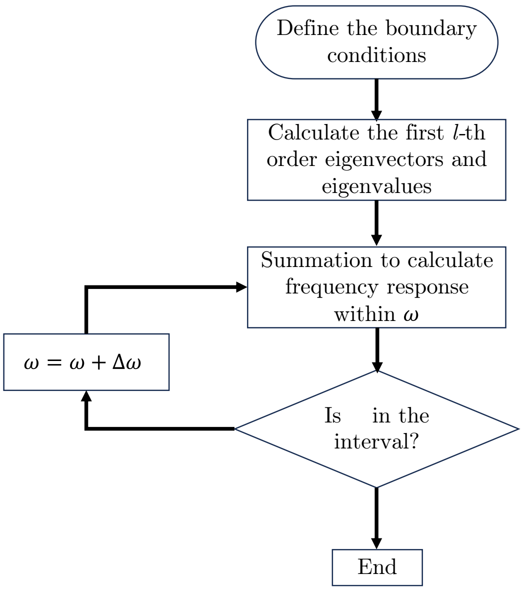


Fig. 3 Process of MDM

Taking a two-dimensional cantilever beam with 6642 degrees of freedom as an example, with one edge subjected to uniformly distributed harmonic excitation as shown in Figure 4. The material parameters are Young's modulus E = 2e11 Pa, Poisson's ratio ν = 0.33, and density ρ = 7890 kg/m^3. The frequency responses in the angular frequency range [0, 5000] are calculated using both methods. The response curve obtained using the FM method is shown in Figure 5 with blue lines, while the MDM results are shown in Figure 5 with red lines. Here, l indicates different orders of eigenvalues and eigenvectors. Comparing the graphics, it can be observed that when the total degrees of freedom of the structure are 6642, if l is greater than or equal to 3, approximately accurate results can be obtained.

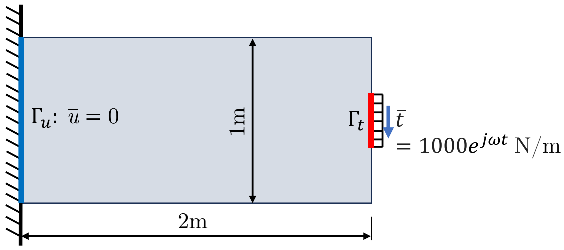
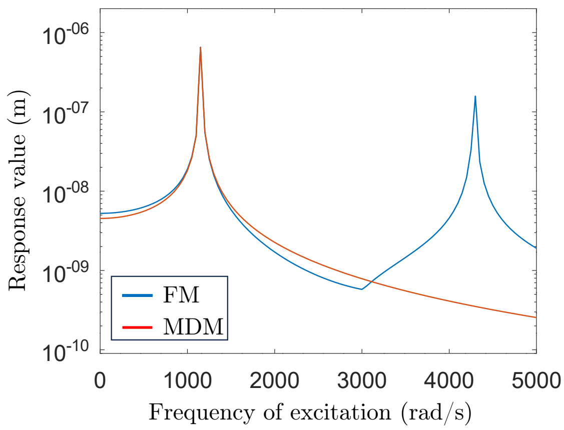
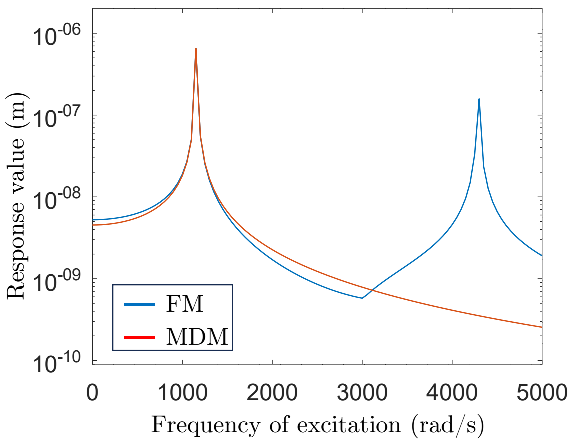


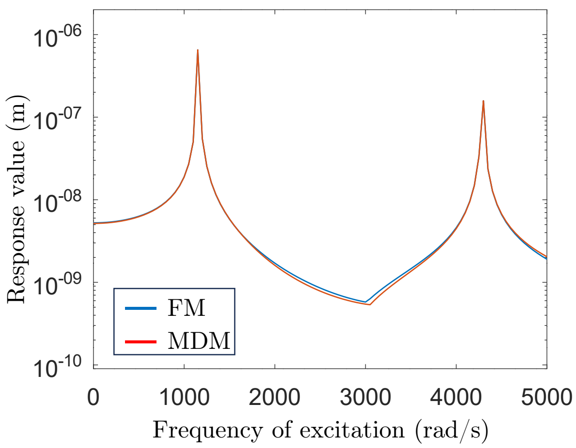
Fig. 4 2D cantilever beam



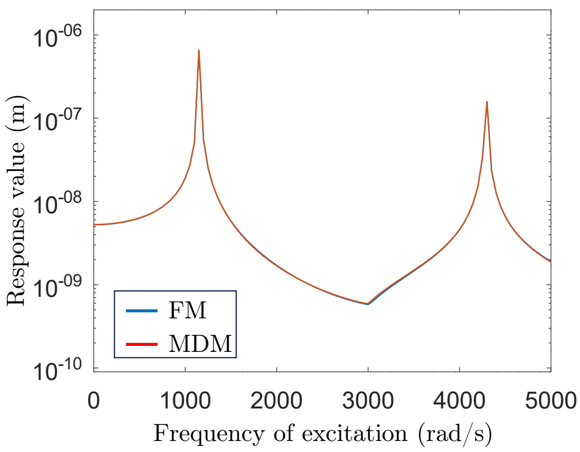
(a) $l = 1$



(b) $l = 2$



(c) $l = 3$



(d) $l = 5$

Fig. 5 When l takes 1, 2, 3 and 5 respectively, the comparison of MDM and FM calculation results. The blue lines are FM, and the red lines are MDM. When $l >= 3$, MDM can get approximately accurate results.

Considering the example of 2D cantilever beam with 420 degrees of freedom, we compare the accuracy achieved with different numbers of eigenvalues and eigenvectors used in MDM. In comparison with the FM method, the relationship between the maximum relative residual of the obtained results and the number is shown in Figure 6. When the number of eigenvectors and eigenvalues l is greater than 100, the relative residual errors r are typically less than 1%, and when $l > 400$, $r = 0$. The calculation formula for r is as follow:

$$r = \max\left(\frac{|\mathbf{u}_{\mathrm{MDM}} - \mathbf{u}_{\mathrm{FM}}|}{\mathbf{u}_{\mathrm{FM}}}\right). \tag{21}$$

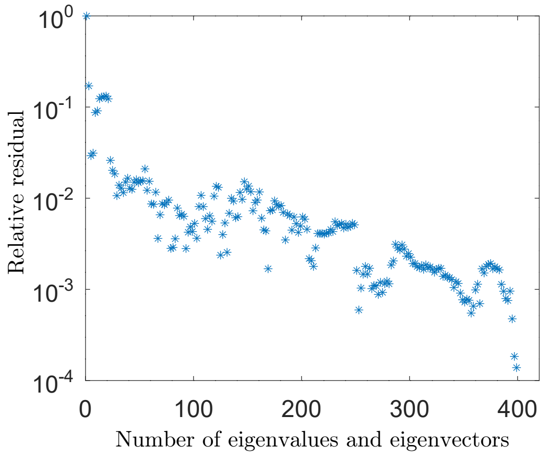


Fig. 6 Relationship between the maximum relative residual of the obtained results and the number of selected modes.

Compare the computation time of the two methods. We use m to represent the number of samples in the frequency interval, t_0 to represent the calculation time of FM to calculate a certain frequency response. The total time of frequency response in the calculation interval is $T_0 = m \cdot t_0$. For MDM, we use t_1 to represent the time required for calculate the first few modes, t_2 to represent the time required to superpose for response at a certain frequency. In this way, the total time for using MDM to calculate the frequency response in the interval is $T_1 = t_1 + m \cdot t_2$. Using the cantilever beam mentioned above as the example. The MDM uses the first 10 order modes, and the single frequency response is calculated in two methods. t_0 is 0.003s. t_1 is 0.005s and t_2 is less than 0.001s. So for a single frequency, $t_1 + t_2 > t_0$, FM requires less calculation time. For the response in frequency range, if the number of frequency samples is m=500, then the total time of FM is T_0 = 2.5s theoretically, and the total time of MDM is T_1 = 0.505s. Therefore, MDM can significantly reduce calculation time. The selection of the number of modes is also affect the total time in MDM. Figure 8 shows in MDM, the relationship between the total calculation time and the number of modes. It can be seen that they are approximately linear relation. When MDM uses all of modes, the time is similar to FM.

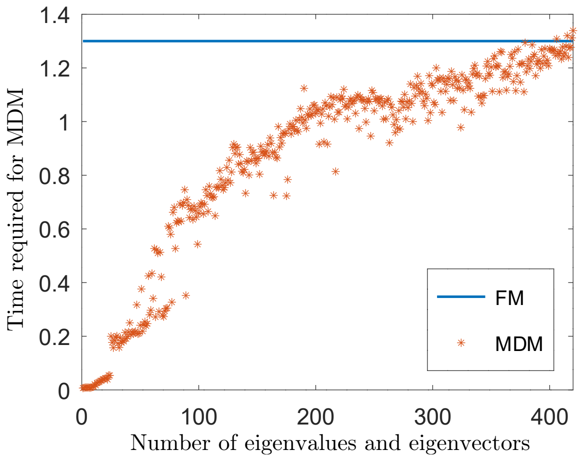


Fig. 7 Computation time of MDM as the number of eigenvectors and eigenvalues increase.

3. Topological derivative

To find the best material distribution in the design domain, we define a level-set function $\phi(x)$ to continuously evolve the new boundary. The updating of the level-set function can be done through the following reaction-diffusion equation:

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = K(-\mathcal{T} + \tau \nabla^2 \phi) & \text{in D} \\ \dfrac{\partial \phi}{\partial n} = 0 & \text{on } \partial \mathrm{D} \backslash \partial \Gamma_t \\ \phi = 1 & \text{on } \partial \Gamma_t \end{cases}, \tag{22}$$

where t is the fictitious time, K is the coefficient of proportionality, τ is a regularization parameter for the fictitious interface energy, D represents the design domain. $\mathcal{T}$ is so-called topological derivative, which determines whether the material should exist at each node, and the evolution direction of the level-set function boundary.

In this paper, topological derivatives are performed to make the sensitivity analysis. The topological derivatives represent the effect of a small hole appearing in the structure on the growth or decrease of the objective function. As shown in Figure 5, when a small hole Ω_ϵ with a radius ϵ appears in the domain Ω, the objective functional J undergoes a change δJ. Therefore, the influence of the hole on the objective function can be expressed as:

$$\mathcal{T} = \lim_{\epsilon \to 0} \frac{(J + \delta J) - J}{f(\epsilon)} = \lim_{\epsilon \to 0} \frac{\delta J}{f(\epsilon)}, \tag{23}$$

where $f(\epsilon)$ is a positive function that $f(\epsilon) \to 0$ when $\epsilon \to 0$. J is the objective functional.

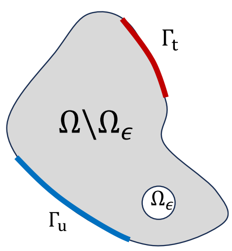


Fig. 8 Design domain Ω generate a small hole Ω_ϵ

The topological derivatives can be calculated by using the adjoint method. For a single frequency boundary integral objective functional:

$$F = \int_\Gamma f(u_i, t_i) \mathrm{d}\Gamma, \tag{24}$$

according to references[(11)(13)], the topological derivative for it can be calculated by:

$$\mathcal{T}_\omega = \frac{3(1-v)}{2(1+v)(7-5v)} \left\{ \frac{-(1-14v+15v^2)E}{(1-2v)^2} \delta_{ij}\delta_{kl} + 5E\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \right\} \tilde{u}^0_{i,j} u^0_{k,l} - \rho\omega^2 \tilde{u}^0_i u^0_i, \tag{25}$$

where $\tilde{u}_i$ is the adjoint operator. When the problem is non-self-adjoint, it can be obtained by solve the problem:

$$C_{ijkl} u_{k,lj} + \rho\omega^2 \tilde{u}_i = 0 \qquad \text{in } \Omega \tag{26}$$

$$\tilde{t}_i = \frac{\partial f(u_i, t_i)}{\partial u_i} \qquad \text{on } \Gamma_t \tag{27}$$

$$\tilde{u}_i = -\frac{\partial f(u_i, t_i)}{\partial t_i} \qquad \text{on } \Gamma_u, \tag{28}$$

which also can be solved by MDM in adjoint field. Through this, it can greatly improve the efficiency of topology optimization. For optimization over a frequency band, the objective function can be defined as a numerical integration over a frequency interval:

$$J = \int_{\omega_1}^{\omega_2} F \mathrm{d}\omega. \tag{29}$$

Correspondingly, the topological derivative also requires numerical integration over the frequency interval:

$$\mathcal{T} = \int_{\omega_1}^{\omega_2} \mathcal{T}_\omega \mathrm{d}\omega. \tag{30}$$

To compare the accuracy of MDM in calculating topological derivatives, we use the aforementioned cantilever beam once again. We select a node on the beam, and then calculate the corresponding topological derivatives of the compliance objective. The frequency response interval is [100, 600]. The relative residual errors between MDM and FM are shown in Figure 9. It can be seen that the majority of the residuals are less than 1.5%. Moreover, when the number of modes exceeds 200, the residual is less than 0.5%.

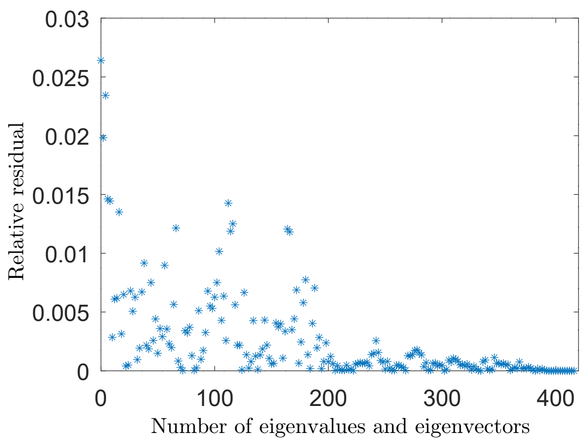


Fig. 9 Relation between the relative residual of the topological derivative and the number of selected modes.

4. Numerical examples

In this section, optimization examples based on MDM and topological derivatives are presented. Also, they are compared with results obtained by FM under the same conditions. For the 2D cantilever beam, in design domain D, we look for the material distribution that minimizes the objective function J, as shown in Figure 10. The material parameters and boundary conditions remain unchanged, and the volume fraction is set to 0.4. The objective function J is the minimum compliance in the frequency interval:

$$J = \int_{\omega_1}^{\omega_2} \int_{\Gamma_t} \bar{t}_i u_i \mathrm{d}\Gamma \,\mathrm{d}\omega \,, \tag{31}$$

where $\omega_1 = 100$, $\omega_2 = 300$. The mesh of the structure contains 16,186 degrees of freedom. In MDM, the number of eigenvectors and eigenvalues is $l = 200$. The regularization parameter τ is equal to 7e-4. Figures 11 and 12 show the results of the two methods. The structures after 50 iterations of optimization are shown in Figures 11(a) and 12(a). The changes in their respective objective functions and volume fractions with each iteration are presented in Figures 11(b) and 12(b), the time required for each optimization iteration step of FM is about 35.29s, and MDM is about 19.60s ($\Delta\omega$=1). Figure 13 shows the results in the frequency interval [100, 500], with all other parameters unchanged, the time required for each optimization iteration step of FM is about 69.63s, and MDM is about 34.39s ($\Delta\omega$=1).

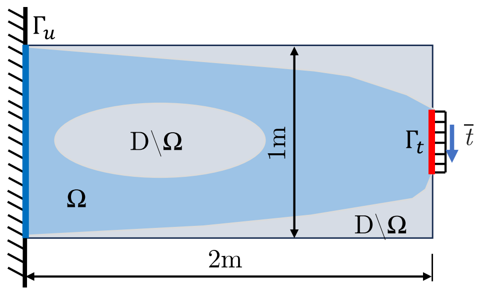
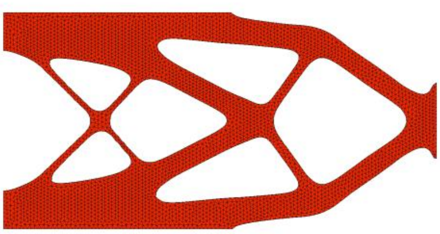
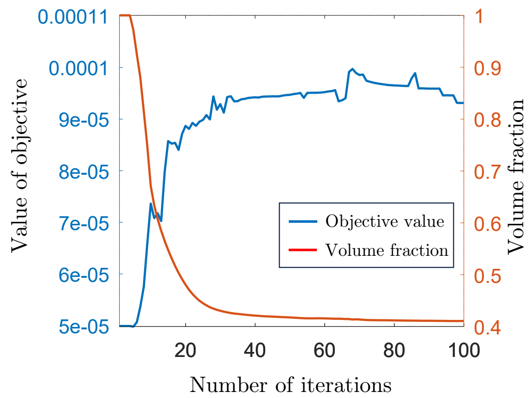


Fig. 10 Example of 2D cantilever beam. D is design domain, **Ω** represents the material area, D**Ω** represents the void domain.

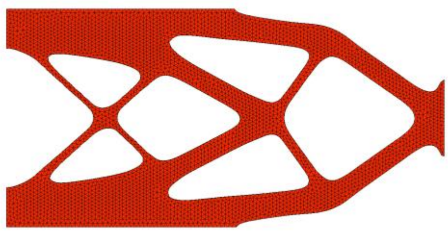


(a) Results after 50 iterations

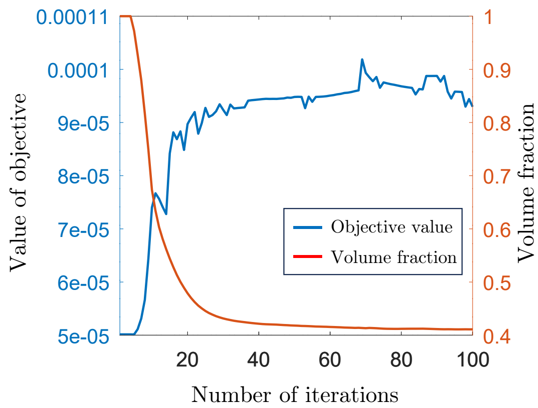


(b) Changes of objective function values and volume fractions

Fig. 11 Result of MDM in frequency interval [100, 300].

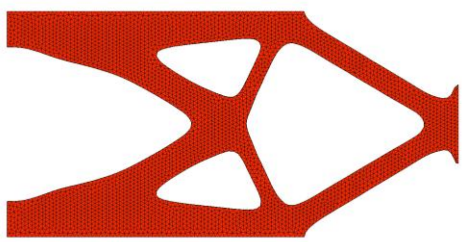


(a) Results after 50 iterations

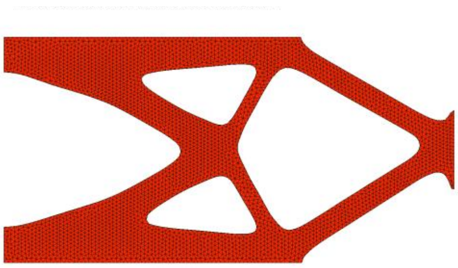


(b) Changes of objective function values and volume fractions

Fig. 12 Result of FM in frequency interval [100, 300].



(a)MDM



(b)FM

Fig. 13 Results of MDM and FM in frequency interval [100, 500].

The results of MDM are basically same as FM, with slight differences. The changes of corresponding objective functions and volume fractions are also roughly same. Therefore, using MDM make the topology optimization can obtain basically accurate results.

5. Conclusion

In calculation of frequency response in intervals, whether it is calculating the frequency response or the adjoint variable of topological derivative, campare with the full mode method, using the mode displacement method can effectively reduce the computational cost and save calculation time.

The results in Figure 12 and Figure 13 are slightly asymmetric. The possible reason is that we used the triangular elements, which lead to mesh asymmetry. Another reason may be the error caused by mode reduction, which is also the direction for improving this method in the future.

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