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A study on eigensolutions of 2D elastic problem by using BEM and FEM

Haifeng GAO[1)], Toru TAKAHASHI[2)], Hiroshi ISAKARI[3)], and Toshiro MATSUMOTO[4)]

1)Nagoya University (Furo-cho, Chikusa-ku, Nagoya, 464-8603, Japan, E-mail: gao_h@nuem.nagoya-u.ac.jp)
2)Nagoya University (Furo-cho, Chikusa-ku, Nagoya, 464-8603, Japan, E-mail: ttaka@nuem.nagoya-u.ac.jp)
3)Nagoya University (Furo-cho, Chikusa-ku, Nagoya, 464-8603, Japan, E-mail: isakari@nuem.nagoya-u.ac.jp)
4)Nagoya University (Furo-cho, Chikusa-ku, Nagoya, 464-8603, Japan, E-mail: t.matsumoto@nuem.nagoya-u.ac.jp)

In this paper, a contour integral-based approach, so-called the block Sakurai-Sugiura (SS) method is applied to solve nonlinear eigenvalue problems formulated by the boundary element method (BEM) for 2D time-harmonic elastic vibration problems. The original eigenspace is limited to a smaller eigenspace by employing a contour integral along a Jordan curve on the complex plane. The eigenfrequencies within the Jordan curve are extracted by solving a linear eigenvalue problem formed by two Hankel matrices whose dimensions are extraordinary smaller than the original system matrix. Furthermore, with a small number of boundary elements, BEM yields results with high accuracy. The comparisons between the numerical results obtained by FEM and BEM for a test eigenvalue problem are presented here to demonstrate the correctness and effectivity of proposed approach.

Key Words: elastic wave, eigensolutions, block SS method, boundary element method

1. Introduction

Unlike finite element method (FEM) (1) that gives a sparse banded coefficient matrix, boundary element method (BEM) (2) provides a full nonsymmetric coefficient matrix $\mathbf{A}$. However, the boundary-only discretization makes the numerical model simpler. Furthermore, in general, since there is no domain interpolation, the solutions obtained by BEM with a certain size of element may be more accurate. For eigenvalue analysis, the direct search method (3), which determines the eigenvalues by drawing a profile of $\det[\mathbf{A}(\omega)]$, can be regarded as a pioneer application of BEM to extract eigenvalues ω or f, where $\omega = 2\pi f$, is the circular frequency. The repeated computation $\det[\mathbf{A}(\omega)]$ with respect to ω with its incremental variation leads to a high computing cost. Moreover, it is very difficult to extract multiple eigenvalues or eigenvalues that are close to each other. Also researchers have developed some transform methods such as internal cell method (4), dual reciprocity method (5) and multiple reciprocity method (6).

The emergence of the contour integral method (7, 8) enables to extract the eigenvalue by using boundary element method without transform. In particular, the block Sakurai-Sugiura (SS) method named after Sakurai and Sugiura is proposed for nonlinear eigenvalue problem (9). By using the block SS method, the investigation on eigenvalue problems for 2D and 3D acoustic cavities is reported in the previous research (10, 11).

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In this paper, eigenvalue problem of a 2D elastic structure which has a more complicated situation, is considered. Elastic wave has both transverse and longitudinal polarizations. The coupling of the transverse and longitudinal waves may make eigenmodes complicated. For FEM models, the internal discretization leads to inaccurate results when the number of elements is insufficient and the eigenmodes are complicated. However, BEM does not have any internal discretization, therefore, with the comparable size for boundary discretization, more accurate results are obtained. A numerical example computed by using both FEM software COMSOL and BEM, is presented. The comparison between FEM and BEM models that have comparable size for boundary discretization is carried out. The computing time of BEM combined with the block SS method depends on many parameters for the block SS method, therefore, the discussion about the computing time remains as a future topic.

2. Formulation

2.1. Analysis of nonlinear eigenvalue problem by the block SS method

Recently, a contour integral method, so-called the SS method (7), by which the size of a generalized eigenvalue problem is converted into a rather smaller one by carrying out the contour integral along a closed Jordan curve, is developed. The reduced eigenspace is determined by the eigenvalues located inside the Jordan curve. Degenerate eigenvalues resulted by symmetry of structure and constraint can be extracted by multi-initial nozero vectors. For nonlinear eigenvalue problem, the block SS method are derived starting from Smith normal form, similarly, the eigenspace of the original problem is reduced to the one of interest and the nonlinear property is eliminated through solving a generalized eigenvalue problem of a linear matrix pencil as follows

$$\mathbf{H}^{<}_{Kl} - \lambda \mathbf{H}_{Kl} = 0 \tag{1}$$

where $\mathbf{H}_{Kl}$ and $\mathbf{H}^{<}_{Kl}$ are Hankel matrices which have been presented in [9] as follows

$$\mathbf{H}_{Kl} = \begin{pmatrix} \mathbf{M}_0 & \mathbf{M}_1 & \cdots & \mathbf{M}_{K-1} \\ \mathbf{M}_1 & \mathbf{M}_2 & \cdots & \mathbf{M}_K \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{K-1} & \mathbf{M}_K & \cdots & \mathbf{M}_{2K-2} \end{pmatrix} \tag{2}$$

$$\mathbf{H}^{<}_{Kl} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{M}_2 & \cdots & \mathbf{M}_K \\ \mathbf{M}_2 & \mathbf{M}_3 & \cdots & \mathbf{M}_{K+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_K & \mathbf{M}_{K+1} & \cdots & \mathbf{M}_{2K-1} \end{pmatrix} \tag{3}$$

where the moment matrices $\mathbf{M}_m$ are defined as as follows

$$\mathbf{M}_m = \frac{1}{2\pi \mathrm{i}} \oint_P \mathbf{U}^H \mathbf{A}^{-1}(z) \mathbf{V} z^m \mathrm{d}z, \tag{4}$$

where $(\cdot)^H$ denotes the conjugate transpose, $\mathbf{A}$ is the system matrix of a nonlinear eigenvalue problem $\mathbf{A}(\omega)\{\mathbf{x}\} = \{\mathbf{0}\}$, P is a positively oriented closed Jordan curve on the complex plane, z is defined on the complex plane with respect to circular frequency ω, $\mathbf{V}$ is a $2N \times l$ matrix formed by column vectors $\mathbf{v}_1$, $\mathbf{v}_2$,..., $\mathbf{v}_l \in \mathbb{C}^{2N}$, and $\mathbf{U} = \mathbf{V}$. Obviously, the number of eigenvalues which can be extracted is determined by Kl (the dimension of the Hankel matrices).

The block SS method also provides the recovery of the eigenvectors, let $\mathbf{w}_j$ be the eigenvectors obtained by Eq. (1), then the eigenvectors $\mathbf{x}_j$ of original eigenvalue problem in Eq. (16) can be obtained by

$$\mathbf{x}_j = \mathbf{S}\mathbf{w}_j \tag{5}$$

where $\mathbf{S} = [\mathbf{S}_0, \mathbf{S}_1, ..., \mathbf{S}_K]$ and $\mathbf{S}_p$ is a intermediate result as follows

$$\mathbf{S}_p = \frac{1}{2\pi \mathrm{i}} \oint_P z^p \mathbf{A}^{-1}(z) \mathbf{V} \mathrm{d}z \tag{6}$$

where the contour integral is carried out by trapezoidal rule numerically.

2.2. Boundary element method for elastodynamic problem

Considering an elastic wave propagating in a homogeneous and isotropic medium without body force, the expression of governing continuum equation by displacements is

$$(C_1^2 - C_2^2) u_{j,jk}(x,t) + C_2^2 u_{k,jj}(x,t) = \ddot{u}_k(x,t) \tag{7}$$

where C_1 and C_2 are the P (longitudinal) wave speed and S (transverse) wave speed respectively, written as,

$$C_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} \tag{8}$$

$$C_2 = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{2\rho(1+\nu)}} \tag{9}$$

where ρ is the density of the medium, λ and μ are lamé constants, E is the young's modulus, and ν is the Poisson' ratio.

For free vibration, no excitation is considered, hence displacements can be written in a time-independent form:

$$u_i(x,t) = U_i(x,\omega) e^{\mathrm{i}\omega t} \tag{10}$$

where i denotes the imaginary unit, $\omega = 2\pi f$ is the circular frequency, and f is the frequency.

Substituting Eq. (10) to Eq. (7), we obtain the time-independent form of governing equation as follows,

$$(C_1^2 - C_2^2) U_{j,jk}(x,\omega) + C_2^2 U_{k,jj}(x,\omega) + \omega^2 U_k(x,\omega) = 0 \tag{11}$$

The boundary integral equation corresponding to the above boundary value problem is obtained as

$$c_{kl}(y) U_k(y,\omega) + \int_\Gamma t^*_{kl}(x,y,\omega) U_k(x,\omega) \mathrm{d}\Gamma(x) - \int_\Gamma u^*_{kl}(x,y,\omega) T_k(x,\omega) \mathrm{d}\Gamma(x) = 0 \tag{12}$$

where c_{kl} depends on the geometry of the boundary at point y, the kernels $u^*(x,y)$ and $t^*(x,y)$ are known as displacement and traction fundamental solutions which are given for two-dimensional case by [2]

$$u^*_{ij}(x,y) = \frac{1}{\alpha\pi\rho C_2^2} \left[\psi \delta_{ij} - \chi r_{,i} r_{,j} \right] \tag{13}$$

where

$$\psi = K_0\left(\frac{sr}{C_2}\right) + \frac{C_2}{sr}\left[K_1\left(\frac{sr}{C_2}\right) - \frac{C_2}{C_1} K_1\left(\frac{sr}{C_1}\right)\right]$$

$$\chi = K_2\left(\frac{sr}{C_2}\right) - \frac{C_2^2}{C_1^2} K_2\left(\frac{sr}{C_1}\right)$$

where $s = \mathrm{i}\omega$, $\alpha = 2$ for 2D case, and K_0, K_1, and K_2 are the modified Bessel function of order 0, 1, and 2, respectively.

Discretizing Eq. (12) with N piecewise constant boundary elements, we obtain a linear equation as follows

$$c_{kl}(y) U^i_k(y,\omega) + \sum_{j=1}^{N} \int_{\Gamma_j} t^*_{kl}(x,y,\omega) \mathrm{d}\Gamma(x) U^j_k(x,\omega) - \sum_{j=1}^{N} \int_{\Gamma_j} u^*_{kl}(x,y,\omega) \mathrm{d}\Gamma(x) T^j_k(x,\omega) = 0 \tag{14}$$

where c_{kl} is 1/2 when the boundary is smooth, and $U^j_k(x,\omega)$, $T^j_k(x,\omega)$ denote the k-direction displacement and traction of the boundary element Γ_j.

Collecting Eq. (14) with source point y on each element, then we obtain a $2N$ system equations as

$$\mathbf{B}\{\mathbf{U}\} = \mathbf{G}\{\mathbf{T}\} \tag{15}$$

where $\mathbf{B}$, $\mathbf{G}$ are $2N \times 2N$ matrices, and vectors $\{\mathbf{U}\}$, $\{\mathbf{T}\}$ $\in \mathbb{C}^{2N}$.

For free vibration of a structure, homogenous boundary condition are usually given on the boundary of the structure.

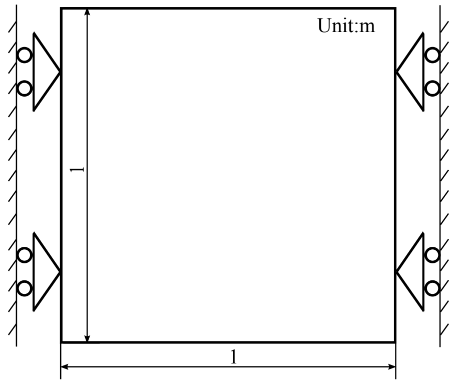


Fig. 1 The 2D elastic square structure.

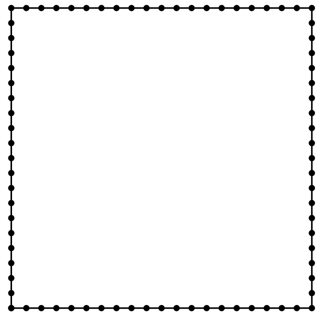


Fig. 2 The mesh of the square structure with 80 uniform elements and 20 elements for each edge (solid points denote the edge nodes of elements).

Therefore, moving the unknowns to the left hand side, we obtain a nonlinear eigenvalue problem with a zero vector in right hand side

$$\mathbf{A}(\omega)\{\mathbf{x}\} = \{\mathbf{0}\} \tag{16}$$

where ω is implicitly involved in each element of $\mathbf{A}$ as an eigenvalue parameter, and $\mathbf{x}$ is the unknown vector.

To solve the nonlinear eigenvalue problem in Eq. (16) formulated by BEM, we have the difficulties on computing a transcendental eigen equation of ω by using standard solver. In this paper, we employ the block SS method to solve Eq. (16).

3. Numerical example

Seeing Fig. 1, a 2D plane-strain square elastic structure with edge length of 1[m] is constrained as depicted. We give a fictitious material with density $\tilde{\rho} = 1.65 \times 10^3$[kg/m], Young's modulus $E = 7.8 \times 10^6$[Pa], and Poisson's ratio $\mu = 0.47$. The boundary is discretized into 80 constant boundary elements (BEs) as shown in Fig. 2, and the solid circles denote the edge nodes of elements.

The contour integral path in Eq. (4) is taken as a circular path: $P = \gamma + \rho e^{i\theta}$ (9), with $\gamma = 500$, $\rho = 100$ in a low frequency range. In order to obtain accurate numerical results, we follow the previous researches to specify the other parameters for the block SS method, which are set as follows: the number of points for trapezoidal rule $n_t = 128$ (11), the order of Hankel matrices $K = 4$ and the number of initial nonzero vectors $l = 20$ (8). We compute $f = \omega/2\pi$ in this paper. The numerical results of ω located in the range $[400, 600]$ (for f, the solved range is $[63.6620, 95.4930]$) are obtained and shown in Table 1 in which all the eigenvalues have relatively small imaginary parts. The eigenvalues with small imaginary parts are considered to be real eigenvalues and we only concern about their real parts. The eigenfrequencies of a FEM model with 1032192 finite elements (FEs) are represented by f_i^* which are considered as a benchmark for the relative errors:

$$R_{\text{error}}(f_i) = |\frac{f_i^* - \text{Re}[f_i]}{f_i^*}| \times 100\% \tag{17}$$

In this paper, all the results of FEM are obtained by using COMSOL with linear triangular elements.

Table 1 The obtained eigenfrequencies of the elastic square structure by BEM with 80 elements.

i	$f_i = \omega_i/2\pi$ [Hz]	R_{error} [%]
1	$65.1034 - 7.8603 \times 10^{-3}i$	4.2150×10^{-3}
2	$70.0517 + 5.2203 \times 10^{-3}i$	2.1489×10^{-2}
3	$71.3134 + 9.7553 \times 10^{-3}i$	3.5129×10^{-2}
4	$76.0215 + 4.3083 \times 10^{-2}i$	1.9756×10^{-1}
5	$77.0927 + 4.0130 \times 10^{-2}i$	2.1323×10^{-1}
6	$82.2306 - 2.3893 \times 10^{-3}i$	1.7457×10^{-2}
7	$82.7139 + 4.4080 \times 10^{-4}i$	6.2123×10^{-3}
8	$84.2805 - 2.8069 \times 10^{-3}i$	1.0985×10^{-2}
9	$85.0675 + 1.9373 \times 10^{-2}i$	3.4258×10^{-4}
10	$87.7443 + 7.9595 \times 10^{-3}i$	4.8193×10^{-2}
11	$95.2639 - 1.0016 \times 10^{-2}i$	9.0897×10^{-2}
12	$95.4605 + 6.1568 \times 10^{-2}i$	2.0081×10^{-1}
13	$95.9628 + 5.9699 \times 10^{-2}i$	3.1083×10^{-1}

In Table 2 and Table 3, the results obtained by FEM using 426, 1708, 3882 FEs and BEM using 40, 80, 120 BEs are presented, respectively. In particular, the results shown in Table 3 have relatively small imaginary parts and $\text{Im}(f_i) \leqslant 2.476 \times 10^{-1}$. It is found that f_8 is more accurate than other eigenvalues in the results of FEM, because its eigenmode is simpler than others as shown Fig. 6(b). In the results of FEM with 426 FEs in Table 2, f_8 is smaller than f_5, f_6 and f_7 because the eigemodes corresponding to f_5, f_6 and f_7 are more complicated and the insufficient number of elements for domain discretization makes f_5, f_6 and f_7 larger than closed forms. With comparable size elements and same boundary discretization, BEM extracts the eigenvalues with correct orders. For simplicity, we just show the eigenmodes corresponding to f_1, f_2, f_7 and f_8 in Figs. 3-6, where the circled dots are the centers of BEs and denote the eigemodes with respect to displacements, and in the eigenmodes obtained by FEM, arrows denote the displacement vectors. Comparing with the other eigenmodes, the eigenmodes corresponding to f_8 is more simple.

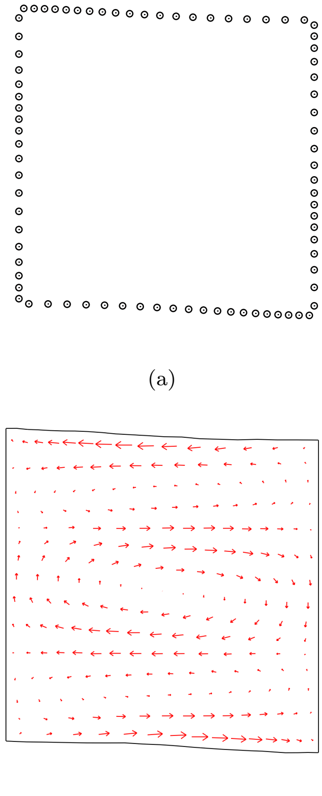
Table 2 The obtained eigenfrequencies f_i of the elastic square structure by FEM software COMSOL with linear triangular elements

i	426 FEs	1708 FEs	3882 FEs
1	68.4500	65.8880	65.4800
2	73.6108	70.9388	70.4495
3	76.2664	72.8448	71.9553
4	83.8080	77.9038	76.8102
5	86.0090	79.3315	77.9867
6	87.8337	83.4917	82.8676
7	89.4310	84.4002	83.5277
8	84.3991*	84.3018*	84.28491
9	92.8805	87.1983	86.0224
10	97.3564	90.5342	88.9014
11	105.4403	97.5653	96.3710
12	108.3386	99.1281	97.0055
13	110.9700	100.0873	97.6078

* The value of f_8 obtained is more accurate than other numerical results. This may have happened due to its simpler eigenmode.

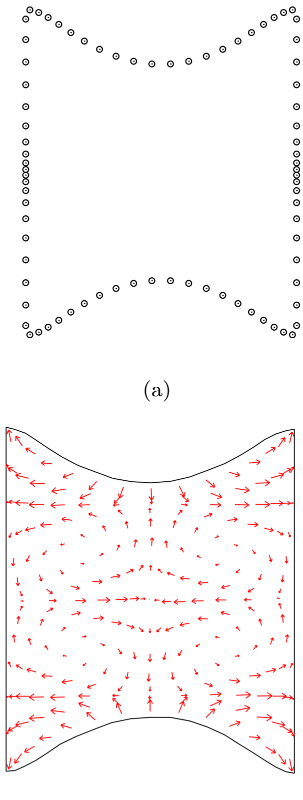
Table 3 The obtained eigenfrequencies f_i of the elastic square structure by BEM and the block SS method

i	40 BEs	80 BEs	120 BEs
1	65.0896	65.1034	65.1037
2	70.0256	70.0517	70.0569
3	71.3845	71.3134	71.2983
4	76.3305	76.0215	75.9502
5	77.4252	77.0927	77.0140
6	82.1752	82.2306	82.2368
7	82.7055	82.7139	82.7137
8	84.2210	84.2805	84.2821
9	85.0806	85.0675	85.0637
10	87.8584	87.7443	87.7193
11	95.1914	95.2639	95.2661
12	96.0996	95.4605	95.3215
13	96.6268	95.9628	95.8166



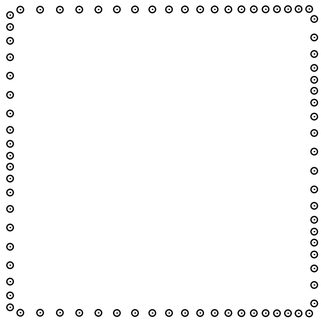
(a) (b)

Fig. 3 The eigenmode corresponding to f_1 obtained by BEM (a) and FEM (b).

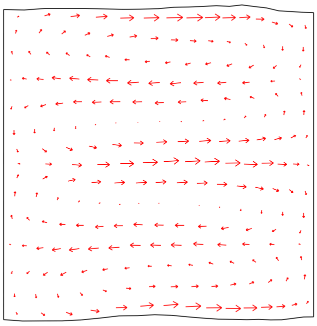


(a) (b)

Fig. 4 The eigenmode corresponding to f_2 obtained by BEM (a) and FEM (b).

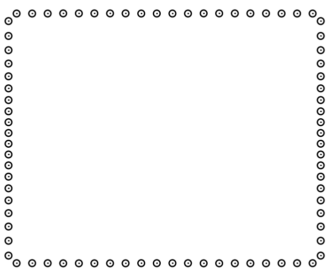


(a)

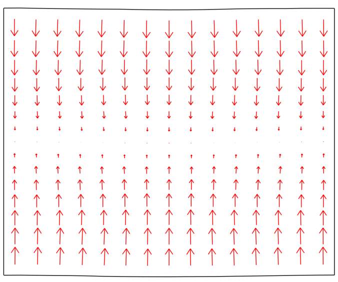


(b)

Fig. 5 The eigenmode corresponding to f_7 obtained by BEM (a) and FEM (b).



(a)



(b)

Fig. 6 The eigenmode corresponding to f_8 obtained by BEM (a) and FEM (b).

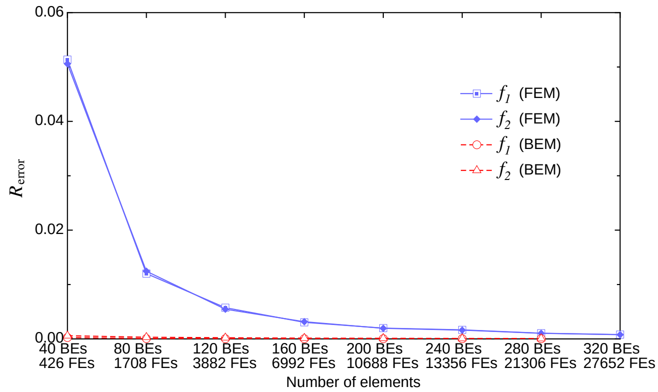


Fig. 7 The relative errors for f_1 and f_2 obtained by FEM and BEM.

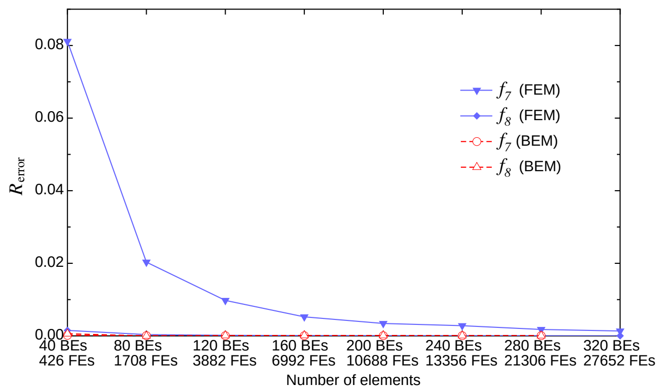


Fig. 8 The relative errors for f_7 and f_8 obtained by FEM and BEM.

The convergences of f_1, f_2, f_7 and f_8 provided by BEM and FEM are shown in Figs. 7 and 8. We use same number of elements on the boundary of BEM and FEM models as 40, 80, 120, 160, 200, 240, 280, 320 BEs, and 426, 1708, 3882, 6992, 10688, 13356, 21306, 27652 FEs. In Fig. 7, f_1 amd f_2 obtained by BEM have $R_{error} \leqslant 5.8678 \times 10^{-4}$(40 BEs), and for the results of FEM, $R_{error} \leqslant 5.136 \times 10^{-2}$(426 FEs). In Fig. 8, f_7 and f_8 obtained by BEM have $R_{error} \leqslant 5.9533 \times 10^{-4}$(40 BEs), and for the results from FEM, f_7 has $R_{error} \leqslant 8.1140 \times 10^{-2}$(426 FEs) while f_8 has smaller $R_{error} \leqslant 1.5183 \times 10^{-3}$(426 FEs) since its eigenmode is simpler than others. It can be seen that the numerical results given by BEM have much smaller relative errors when eigenmodes are complicated.

4. Conclusions

The investigation on eigensolutions for 2D elastic structure is carried out by using BEM combined with block SS method, and the results are compared with those obtained by FEM. With the same boundary discretization for BEM and FEM, BEM provides more accurate numerical results since it does not have domain discretization. For FEM, it is found that a insufficient domain discretization lowers the accuracy of the eigenvalues when the eigenmodes in internal domain are complicated, therefore, the order of eigenvalues might be different from the closed form. However, the results obtained by BEM are not affected by the complexity of eigenmodes in internal domain.

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