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Eigenvalue analysis for 2D acoustic problem by BEM with block SS method

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This paper uses BEM and the block Sakurai-Sugiura Method (block SS method) to solve the eigenvalue problems governed by two-dimensional Helmholtz equation. The rank test is required in the block-version of the algorithm of SS method. A threshold is employed for filtering out the small singular values obtained by singular value decomposition (SVD). The singular values smaller than its threshold can be neglected, while in some cases with different parameters for block SS method, the singular values that should be neglected become comparatively large. A behavior of singular values that can be cut off is investigated by numerical experiments.

Key Words: Eigenvalues, Helmholtz Equation, BEM, Block Sakurai-Sugiura Method

1. Introduction

The boundary element method (BEM) has not been a suitable numerical method to solve eigenvalue problems obtained from the Helmholtz equation and other similar differential equations, because its high storage requirement and the nonlinearlity of the eigenvalue problem. However, the appearance of the fast multipole BEM (FMBEM)[1] has resolved the storage problem, as well as the high computation cost for large scale problems. Also, contour integral methods for nonlinear eigenvalue problem[2, 3, 4, 5, 6, 7] has been actively investigated recently. Hence, the BEM may also be used effectively for nonlinear eigenvalue problems now by combining both approaches.

The BEM requires the fundamental solution of the Helmholtz equation for analyses. The parameter, wave number k, for which eigenvalues are obtained, are involved in the transcendental functions of the fundamental solutions. The components of the coefficient matrix related to the discretized boundary integral equation are calculated by integrating the fundamental solution for each element, thus the coefficient matrix of the system of linear algebraic equations implicitly involves the wave number in its components. Therefore, we could not use the standard eigenvalue solvers for the nonlinear eigenvalue problem resulting from using BEM.

Recently, methods based on contour integral have been proposed for the solutions of nonlinear eigenvalue problems and been actively applied to various problems. They can extract the eigenvalues located in a certain region enclosed with a contour of the complex plane. Also, one version of the methods, the block SS method[4], can give the multiplicities of the eigenvalues at the same time.

In this paper, we treat a plane acoustic problem governed by the Helmholtz equation and try to calculate the eigenvalues by using BEM and the block SS method. Singular values of the Hankel Matirices are used for their rank tests to determine the number of eigenvalues included in the defined contour on the complex plane. The behavior of the singular values versus N of the N-point trapezoidal rule that evaluate the path integral is studied. The threshold used to distinguish between the large singular values and negligible singular values is sought. Numerical examples are shown to demonstrate the overall procedure.

2. Formulation

We consider a plane acoustic problem in time-harmonic vibration as the physical phinomenon governed by the Helmholtz equation:

$$\nabla^2 p(x) + k^2 p(x) = 0 \quad \text{in } D, \tag{1}$$

where $p(x)$ is the sound pressure at a point x in a homogeneous domain D in the plane, ∇^2 is the Laplace's operator, and k is the wave number.

Equation (1) can be transformed to an integral representation by using the following fundamental solution (2) of the two-dimensional Helmholtz equation:

$$p^*(x, y) = \frac{i}{4} H_0^{(1)}(kr), \tag{2}$$

where x and y are two different points in D, r is the distance between x and y, and $H_0^{(1)}$ denotes the zero order Hankel function of

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the first kind.

Then, the sound pressure at an arbitrary point in D can be related to the sound pressure and its normal derivative on the boundary by

$$p(y) = \int_S p^*(x, y)\frac{\partial p}{\partial n}(x)dS(x) - \int_S \frac{\partial p^*(x, y)}{\partial n(x)} p(x)dS(x), \tag{3}$$

where n denotes the outward normal to the boundary. The boundary integral equation is obtained by taking the limit $y \in D \to y \in S$, as follows:

$$c_y p(y) + \int_S \frac{\partial p^*(x, y)}{\partial n(x)} p(x)dS(x) = \int_S p^*(x, y)\frac{\partial p}{\partial n}(x)dS(x), \tag{4}$$

where $c_y = \frac{1}{2}$ if y is located at a smooth part of the boundary S.

Equation (4) can be converted to a regularized form, which is used in the numerical computations in this paper, as follows:

$$\begin{aligned}&\int_S \frac{\partial p^*(x, y)}{\partial n(x)} [p(x) - p(y)]\, dS(x)\\ &\quad + \left\{\int_S \left[\frac{\partial p^*(x, y)}{\partial n(x)} - \frac{\partial P^*(x, y)}{\partial n(x)}\right] dS(x)\right\} p(y)\\ &\quad = \int_S p^*(x, y)\frac{\partial p}{\partial n}(x)dS(x),\end{aligned} \tag{5}$$

where $P^*(x, y)$ is the fundamental solution of Laplace's equation.

Applying the the homogenous boundary condition and using quadratic isoparametric element for discretizing Eq.(5), we obtain a system of linear algebraic equations, as follows:

$$\mathbf{A}(k)\mathbf{x} = 0, \tag{6}$$

where $\mathbf{A}(k)$ is a matrix whose components are obtained by evaluating the integrals either of p^* or $\partial p^*/\partial n$, and $\mathbf{x}$ is a column vector comprising unknown values at the nodes on the boundary.

The Block SS method can extract the eigenvalues of the nonlinear eigenvalue problem (6) lying inside a Jordan curve Γ on the complex plane, and can keep the multiplicity of the eigenvalues less than l that is the number of the column of the arbitrary nonzero matrices $\mathbf{V}$ and $\mathbf{U}$, which are used for computation of the moment matrices

$$\mathbf{M}_m = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathbf{U}^H \mathbf{A}(z)^{-1}\mathbf{V}z^m \mathrm{d}z, \tag{7}$$

where in the present paper we take $\mathbf{U} = \mathbf{V}$, and 'i' is the imaginary unit. This contour integral can be evaluated by using N-point trapezoidal rule, then two Hankel matrices $\mathbf{H}^<_{Kl}$ and $\mathbf{H}_{Kl}$ can be constructed by the moment matrices (7)

$$\mathbf{H}_{Kl} = [\mathbf{M}_{i+j-2}]^K_{i,j=1} \tag{8}$$

$$\mathbf{H}^<_{Kl} = [\mathbf{M}_{i+j-1}]^K_{i,j=1}, \tag{9}$$

where $\mathbf{M}_{i+j-2}$ and $\mathbf{M}_{i+j-1}$ are $l \times l$ $i + j - 2$ and $i + j - 1$ order moment matrices respectively, $i, j = 1, 2, \cdots, K$. In SS method K is the dimension of Hankel matrices and in the block version the dimension of Hankel matrices is Kl because moment becomes a $l \times l$ matrix.

Therefore, by solving eigenvalues of the matrix pencil:

$$\mathbf{H}^<_{Kl} - k\mathbf{H}_{Kl}, \tag{10}$$

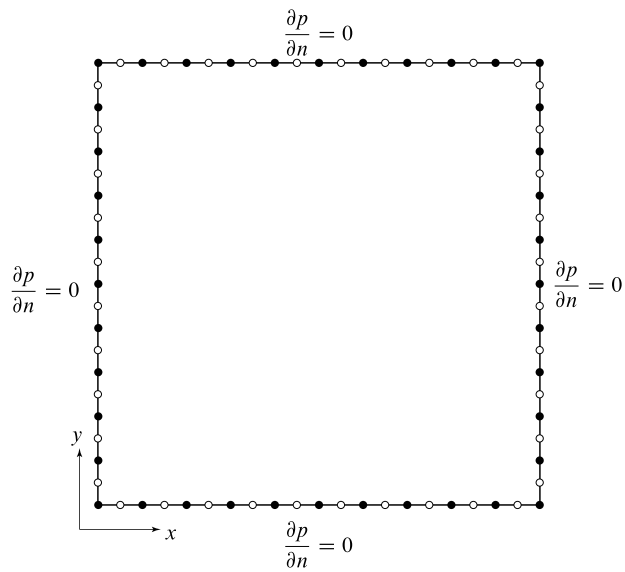

Fig. 1 A square region, the boundary condition, and boundary elements (40 quadratic elements are used. The solid symbols are the edge nodes and the open symbols are the middle nodes of the elements)

we can obtain the original eigenvalues $k_1, k_2, \cdots, k_{Kl}$ contained in the closed curve Γ. After the Hankel matrices being constructed, the singular value decomposition of $\mathbf{H}_{Kl}$ is carried out to make a rank test, as

$$\mathbf{H}_{Kl} = \mathbf{C}\mathbf{E}^H, \tag{11}$$

where $\mathbf{C}$ and $\mathbf{E}$ are complex unitary matrices, is a diagonal matrix with nonnegative real numbers at its diagonal elements.

Then original eigenvalue problem can be cast to a standard one by computing the eigenvalues of the matrix

$$\mathbf{B} = \mathbf{C}^H\mathbf{H}^<_{Kl}\mathbf{E}^{-1}. \tag{12}$$

At the step of SVD, the rank test is carried out, and we attempt to find a suitable threshold value for cutting off sufficiently small, and thus negligible, singular values. The behavior of the singular values against N is also investigated in [8].

3. Numerical example

Consider a simple 2D square domain with sides in 1 [m] as shown in the Fig.1. Neumann boundary condition are assumed on all the edges. The theoretical eigenvalues of this problem are given by

$$k_e = \pi\sqrt{\left(\frac{t_x}{L_x}\right)^2 + \left(\frac{t_y}{L_y}\right)^2}, \quad (t_x = 0, 1, 2, 3, \cdots,\ t_y = 0, 1, 2, 3, \cdots). \tag{13}$$

When $L_x = L_y = 1$, the closed form of the eigenmodes are given by

$$P_e = A\cos(t_x\pi x)\cos(t_y\pi y), \quad (t_x = 0, 1, 2, 3, \cdots,\ t_y = 0, 1, 2, 3, \cdots). \tag{14}$$

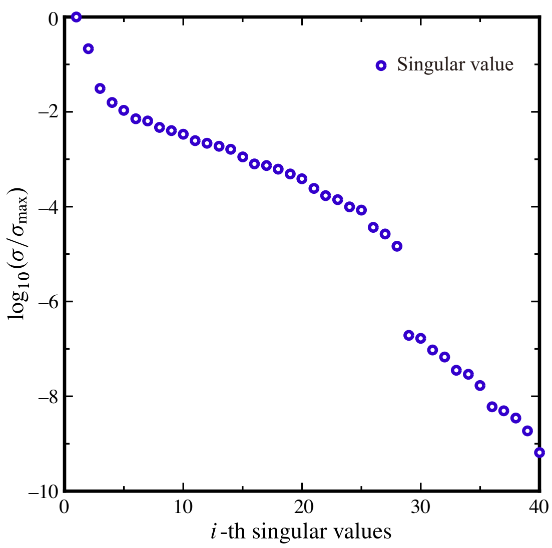

Fig.2 The singular values for fixed parameter $K = 4$, $l = 10$ with $N = 128$.

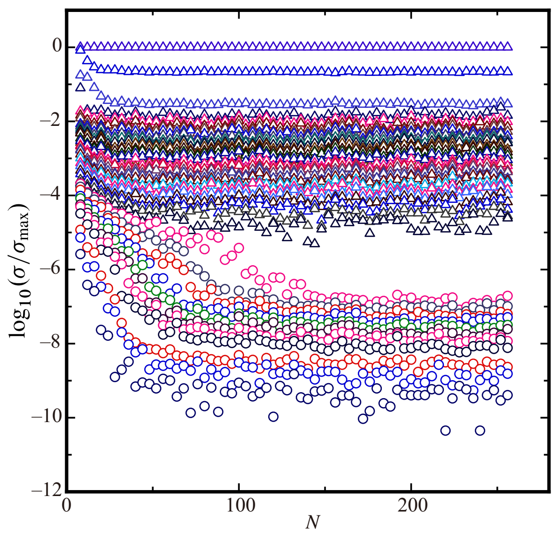

Fig.3 The separation of singular values.

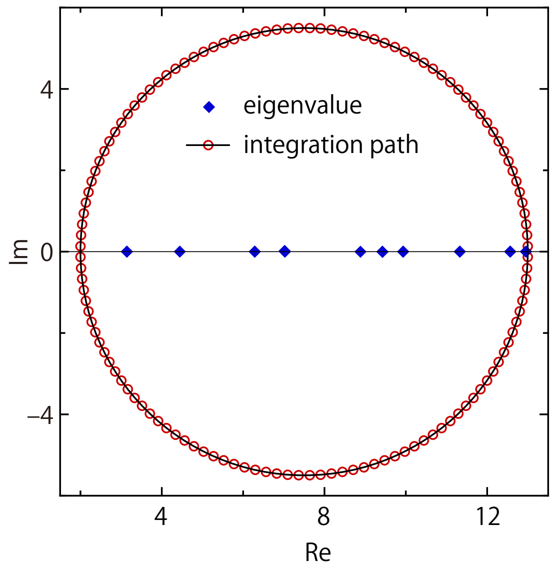

Fig.4 The circular integral path and the obtained eigenvalues.

We discretize the boundary into 40 quadratic isoparametric elements and employ the BEM based on Eq.(5). We calculate the eigenvalues within a circular integral path $\Gamma = \gamma + \rho e^{i\theta}$, where $\gamma = (7.5, 0)$ and $\rho = 5.5$, with $K = 4$ and $l = 10$ that are the parameters of the block SS method. Assuming that there are no rank drops in Hankel matrices, the number of eigenvalues located within the integral path becomes smaller than $K \times l$. In this case, the maximum number of eigenvalues we can calculate within the integral path is 40. We observe in Fig.2 that there is a jump in the magnitude of the normalized singular values between $10^{-5} \sim 10^{-7}$ with $N = 128$. Therefore, we may have singular values that are sufficiently small and can neglect. For a smaller number of N, we do not observe such a gap. Hence, we make an attempt to calculate the normalized singular values by changing N. The matrix pencil (10) is equivalent to the matrix pencil formed by 0-th order and first-order moment[3, 8], and thus, SS method can detect the rank of $\mathbf{H}_{Kl}$ with sufficiently large number for N. Assuming $Kl > m'$, where m' is the number of eigenvalues located within the circular integral path, the behavior of the singular values against N is shown in Fig.3. We find that a separation of the singular values appears, and for $N > 100$ we do not observe the decrease of small singular values, thus we have sufficient gaps between the larger and smaller singular values. In the Figure, the open triangular symbols represent the larger singular values, while the open circular symbols represent the smaller ones. If the threshold δ of the singular value is chosen within the separation range $10^{-5} \sim 10^{-7}$ to filter out the normalized singular values $\sigma/\sigma_{\max}$, the singular values smaller than δ can be cut off. Using the significant singular values obtained in this way, we are able to calculate the eigenvalues. But the eigenvalues of the present problems must always lie on the real axis. Therefore, the eigenvalues having large imaginary part are regarded as ghosts and can be removed from the eigenvalues. In Table 1 are shown the obtained eigenvalues of the wave numbers for $N = 128$ by setting the tolerance threshold value of the imaginary part of the eigenfrequency as 0.05. The number of eigenvalues finally obtained inside the integral path is 18 as shown in the Table. All the values have reasonable accuracy. Fig.4 shows the locations of the obtained eigenvalues in the circular integral path by using rhombus symbols.

The eigen-modes of the interior domain can be computed by using the eigen-pairs obtained by the SS method. The eigenvalues and eigenvectors are substituted into integral representation (3) to calculate the sound pressure amplitude at internal points, and thus obtained eigen-modes corresponding to the eigenvalues $\sqrt{2}\pi$ and $2\sqrt{2}\pi$ are shown in Figs.5 and 6, respectively. Both of the results agree well with the theoretical eigen-modes.

4. Conclusion

The block SS method based on contour integrals has been em-

Table 1 Numerical results of the eigenvalues and relative errors.

i	t_x	t_y	k_i	Relative error [%]
1	1	0	3.1415006	0.00293
2	0	1	3.1418902	0.00947
3	1	1	4.4427825	0.00226
4	2	0	6.2832447	0.00095
5	0	2	6.2876869	0.07165
6	2	1	7.0205457	0.06077
7	1	2	7.0269803	0.03083
8	2	2	8.8844560	0.01474
9	3	0	9.4229509	0.01939
10	0	3	9.4280028	0.03422
11	3	1	9.9334976	0.01098
12	1	3	9.9352245	0.00640
13	2	3	11.3272761	0.00091
14	3	2	11.3287460	0.01388
15	0	4	12.5666751	0.00242
16	4	0	12.5668878	0.00412
17	1	4	12.9536691	0.00425
18	4	1	12.9539116	0.00612

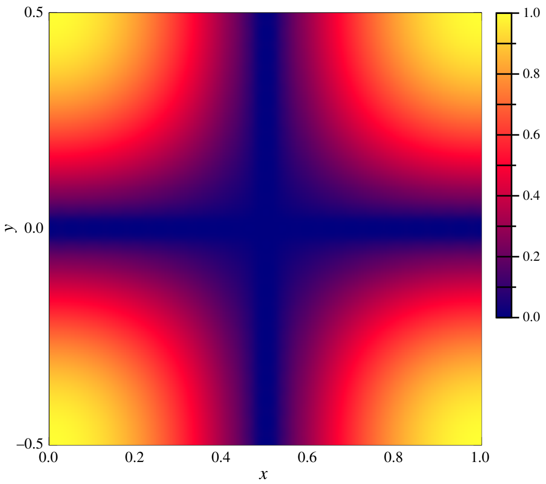

Fig.5 The obtained eigenmode corresponding to $\sqrt{2}\pi$.

ployed to solve nonlinear eigenvalue problems formulated based on BEM for the two-dimensional Helmholtz equation. Sufficiently small singular values of the Hankel matrix are neglected. The threshold of the singular value for filtering out the negligible normalized singular values are determined by investigating the normalized singular values obtained by increasing the number of partitions N used to evaluate the path integral by N-points trapezoidal rule. A numerical example has shown that the present approach can give accurate eigenvalues and eigenvectors.

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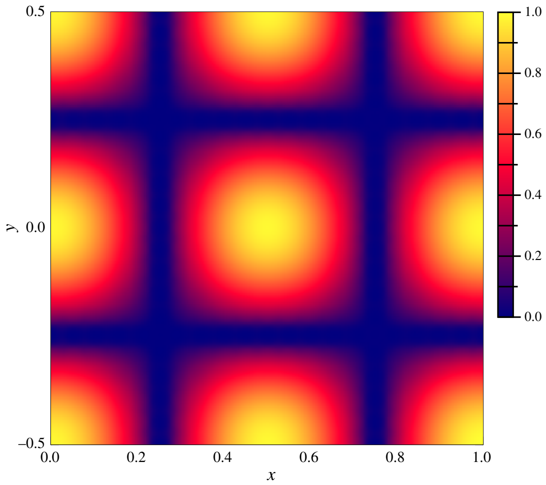

Fig.6 The obtained eigenmode corresponding to $2\sqrt{2}\pi$.

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